2-D Linear Systems of the Form: $\dfrac{d\vec{\mathbf{x}}}{dt} = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ and $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{b}}$

[1] Consider $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 - 3 \end{bmatrix}$.

(a) Find all equilibria.

(b) Find general solutions.

(c) Solve under the initial condition $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) Sketch the phase portrait.

(e) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

[2] Consider $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 0 & 5 \\ -2 & -2 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -5 \\ -2 \end{bmatrix}$.

(a) Find all equilibria.

(b) Convert the system to the form $\dfrac{d\vec{\mathbf{x}}}{dt} = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

(c) Find general solutions.

(d) Sketch the phase portrait.

(e) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

[3] Consider $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

(a) Find all equilibria.

(b) Convert the system to the form $\dfrac{d\vec{\mathbf{x}}}{dt} = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

(c) Find general solutions.

(d) Sketch the phase portrait.

(e) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

Answers:

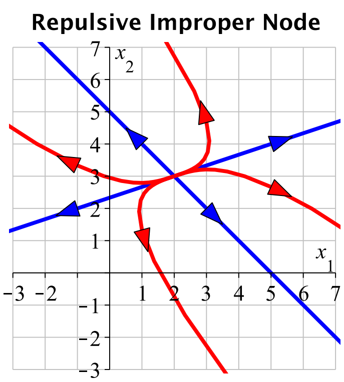
[1] (a) $(2, 3)$.

(b) $\vec{\mathbf{x}}(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where C_1 and C_2 are free parameters.

(c) $\vec{\mathbf{x}}(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2e^{6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

or, equivalently, $\vec{\mathbf{x}}(t) = \begin{bmatrix} 2 - 3e^{2t} + 2e^{6t} \\ 3 - e^{2t} - 2e^{6t} \end{bmatrix}$.

(d)



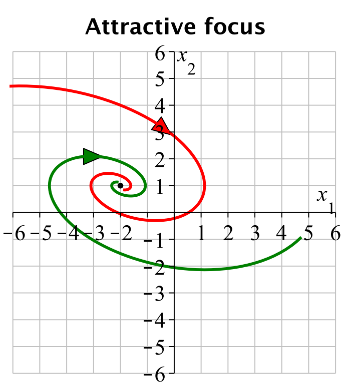
(e) Equilibrium $(2, 3)$ is unstable.

[2] (a) $(-2, 1)$.

(b) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 + 2 \\ x_2 - 1 \end{bmatrix}$.

(c) $\vec{\mathbf{x}}(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_1 e^{-t} \left(\cos(3t) \begin{bmatrix} 5 \\ -1 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + C_2 e^{-t} \left(\sin(3t) \begin{bmatrix} 5 \\ -1 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right)$

(d)



(e) Equilibrium $(-2, 1)$ is asymptotically stable.

[3] (a) All points on the line passing through the point $(0, 1)$ parallel to the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Or, equivalently, all points on the line $x_2 = 1 + \frac{1}{2}x_1$.
Or, we may also say: all points $(x_1, x_2) = (2x_2 - 2, x_2)$, where x_2 is arbitrary.

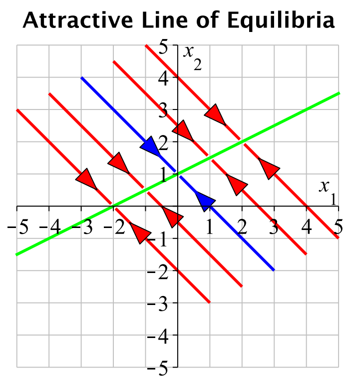
(b) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix}$.

Remark: To convert, I picked a particular equilibrium $(0, 1)$. Any other equilibrium will do as well. For instance, if we pick $(4, 3)$, then the converted system will be $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 - 4 \\ x_2 - 3 \end{bmatrix}$.

(c) $\vec{\mathbf{x}}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Or, $\vec{\mathbf{x}}(t) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is correct too .....

(d)



(e) Every equilibrium is stable but not asymptotically stable.