Higher Order Linear Differential Equations with Constant Coefficients

Part I. Homogeneous Equations: Characteristic Roots

Objectives: Solve n-th order homogeneous linear equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_n, \cdots, a_1, a_0$ are constants with $a_n \neq 0$.

Solution Method:

- Find the roots of the characteristic polynomial:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

- Each root λ produces a particular exponential solution $e^{\lambda t}$ of the differential equation.
- A repeated root λ of multiplicity k produces k linearly independent solutions $e^{\lambda t}, te^{\lambda t}, \cdots, t^{k-1}e^{\lambda t}$.

Warning: The above method of characteristic roots does not work for linear equations with variable coefficients. As matter of fact, the explicit solution method does not exist for the general class of linear equations with variable coefficients.

Example 1: (a) Find general solutions of $y''' + 4y'' - 7y' - 10y = 0$.

(b) Solve $y''' + 4y'' - 7y' - 10y = 0, y(0) = -3, y'(0) = 12, y''(0) = -36$.

Solution: (a) Solve the characteristic polynomial:

$$\lambda^3 + 4\lambda^2 - 7\lambda - 10 = 0.$$

The roots are $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = -5$. Each root gives a particular exponential solution of the differential equation. Combined, the general solutions are

$$y = C_1 e^{-t} + C_2 e^{2t} + C_3 e^{-5t},$$

where C_1, C_2, C_3 are free parameters (arbitrary constants).

(b) We use the initial conditions to determine the values of the constants C_1, C_2, C_3 in the general solution formula. The initial conditions $y(0) = -3, y'(0) = 12, y''(0) = -36$ yield

$$\begin{aligned} C_1 + C_2 + C_3 &= -3, \\ -C_1 + 2C_2 - 5C_3 &= 12, \\ C_1 + 4C_2 + 25C_3 &= -36. \end{aligned}$$

Solve this linear system for C_1, C_2, C_3:

$$\begin{aligned} C_1 &= -5/2, \\ C_2 &= 1, \\ C_3 &= -3/2. \end{aligned}$$

Thus, the solution to the initial value problem is

$$y = -\frac{5}{2}e^{-t} + e^{2t} - \frac{3}{2}e^{-5t}.$$

Example 2: (a) Find general solutions of $y^{(4)} + 8y'' + 16y = 0$.
(b) Solve $y^{(4)} + 8y'' + 16y = 0, y(0) = -1, y'(0) = 5, y''(0) = -8, y'''(0) = -28$.

Solution: (a) Solve the characteristic polynomial:

$$0 = \lambda^4 + 8\lambda^2 + 16 = (\lambda^2 + 4)^2.$$

The roots are $\lambda_1 = 2i, \lambda_2 = 2i, \lambda_3 = -2i, \lambda_4 = -2i$. We have repeated roots. For complex characteristic roots, we can either use complex exponential functions or use cos and sin to express the solutions.

Expression 1: The general solutions are

$$y = C_1 \cos(2t) + C_2 \sin(2t) + C_3 t \cos(2t) + C_4 t \sin(2t),$$

where C_1, C_2, C_3, C_4 are free parameters (arbitrary constants).

Expression 2: The general solutions are

$$y = a_1 e^{2it} + a_2 t e^{2it} + a_3 e^{-2it} + a_4 t e^{-2it},$$

where a_1, a_2, a_3, a_4 are free parameters (arbitrary constants).

(b) Let's use Expression 1 in the above. The initial conditions yield a linear system for $C_1, \cdots, C_4$:

$$\begin{array}{ccccccc} C_1 & & & & = & -1, \\ & 2C_2 & +C_3 & & = & 5, \\ -4C_1 & & & +4C_4 & = & -8, \\ & -8C_2 & -12C_3 & & = & -28. \end{array}$$

Solving this we obtain $C_1 = -1, C_2 = 2, C_3 = 1, C_4 = -3$. Thus, the solution is

$$y = -\cos(2t) + 2\sin(2t) + t\cos(2t) - 3t\sin(2t).$$

We're done.

Alternatively, we may use Expression 2 as well. The initial conditions yield

$$\begin{array}{cccccc} a_1 & & +a_3 & & = & -1, \\ 2ia_1 & +a_2 & -2ia_3 & +a_4 & = & 5, \\ -4a_1 & +4ia_2 & -4a_3 & -4ia_4 & = & -8, \\ -8ia_1 & -12a_2 & +8ia_3 & -12a_4 & = & -28. \end{array}$$

Solving this we obtain $a_1 = -1/2 - i, a_2 = (1 + 3i)/2, a_3 = -1/2 + i, a_4 = (1 - 3i)/2$. Thus, the solution is

$$y = \left(-\frac{1}{2} - i\right) e^{2it} + \left(\frac{1}{2} + \frac{3}{2}i\right) t e^{2it} + \left(-\frac{1}{2} + i\right) e^{-2it} + \left(\frac{1}{2} - \frac{3}{2}i\right) t e^{-2it}.$$

Example 3: Suppose that a 14-th order homogeneous linear differential equation with constant coefficients has characteristic roots:

$$-3, 1, 0, 0, 2, 2, 2, 2, 3+4i, 3+4i, 3+4i, 3-4i, 3-4i, 3-4i.$$

What are the general solutions of the differential equation?

Solution: The general solutions are

$$\begin{aligned} y = {} & C_1 e^{-3t} + C_2 e^{t} + C_3 + C_4 t + C_5 e^{2t} + C_6 t e^{2t} + C_7 t^2 e^{2t} + C_8 t^3 e^{2t} \\ & + C_9 e^{3t}\cos(4t) + C_{10} e^{3t}\sin(4t) \\ & + C_{11} t e^{3t}\cos(4t) + C_{12} t e^{3t}\sin(4t) \\ & + C_{13} t^2 e^{3t}\cos(4t) + C_{14} t^2 e^{3t}\sin(4t), \end{aligned}$$

where $C_1, \cdots, C_{14}$ are free parameters.

Alternative Expression: The general solutions are

$$\begin{aligned} y = {} & C_1 e^{-3t} + C_2 e^{t} + C_3 + C_4 t + C_5 e^{2t} + C_6 t e^{2t} + C_7 t^2 e^{2t} + C_8 t^3 e^{2t} \\ & + C_9 e^{(3+4i)t} + C_{10} t e^{(3+4i)t} + C_{11} t^2 e^{(3+4i)t} \\ & + C_{12} e^{(3-4i)t} + C_{13} t e^{(3-4i)t} + C_{14} t^2 e^{(3-4i)t}, \end{aligned}$$

where $C_1, \cdots, C_{14}$ are free parameters.

Part II. Nonhomogeneous Equations: Undetermined Coefficients

Objectives: Solve n-th order nonhomogeneous linear equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t),$$

where $a_n, \cdots, a_1, a_0$ are constants with $a_n \neq 0$, and $f(t)$ is a given function.

Solution Method:

- The general solutions of the nonhomogeneous equation are of the following structure:

  $$y = y_c + y_p,$$

  where y_c (the so-called "complementary" solutions) are solutions of the corresponding homogeneous equation:

  $$a_n y_c^{(n)} + a_{n-1} y_c^{(n-1)} + \cdots + a_1 y_c' + a_0 y_c = 0,$$

  and y_p is a particular solution of the given nonhomogeneous equation.

- We copy the function structure of the nonhomogeneous term $f(t)$ to set up a trial function for y_p.

- A "naive" choice of the trial function y_p does not work in the case of resonance; that is, we need to modify the choice of y_p in the case when there are overlaps between y_c and y_p. The guiding principle of the modification is to multiply the corresponding term(s) by t^m, where m is the smallest positive integer to make the overlap between y_c and y_p disappear.

Limitations: The method of undetermined coefficients does not work for linear equations with variable coefficients. Also the nonhomogeneous term $f(t)$ is restricted to be of the following forms:

a polynomial $p(t) = a_0 + a_1 t + \cdots + a_N t^N$,
an exponential function e^{at},
$\cos(\omega t)$,
$\sin(\omega t)$,
(a polynomial)$\cdot e^{at}$,
(a polynomial)$\cdot \cos(\omega t)$,
(a polynomial)$\cdot \sin(\omega t)$,
(a polynomial)$\cdot e^{at} \cos(\omega t)$,
(a polynomial)$\cdot e^{at} \sin(\omega t)$,
or a linear combination of the above functions.

For those $f(t)$ that are not one of the above, the method of variation of parameters should be used to solve the nonhomogeneous equation. Indeed, the method of variation of parameters is a more general method and works for arbitrary nonhomogeneous term $f(t)$ (including the types that can be solved by the method of undetermined coefficients).

Example 4: (a) Solve $y''' + 4y'' - 7y' - 10y = 100t^2 - 64e^{3t}$.

(b) Solve $y''' + 4y'' - 7y' - 10y = 100t^2 - 64e^{3t}, y(0) = -20, y'(0) = -29/5, y''(0) = 19/5$.

Solution: (a) First, solve the corresponding homogeneous equation

$$y_c''' + 4y_c'' - 7y_c' - 10y_c = 0.$$

This was done in Example 1. The complementary solutions are

$$y_c = C_1e^{-t} + C_2e^{2t} + C_3e^{-5t},$$

where C_1, C_2, C_3 are free parameters.

Next, set up a trial function by copying the structure of $f(t)$:

$$y_p = a_0 + a_1t + a_2t^2 + be^{3t}.$$

Substitute this into the nonhomogeneous equation and simplify:

$$(-10a_0 - 7a_1 + 8a_2) + (-10a_1 - 14a_2)t - 10a_2t^2 + 32be^{3t} = 100t^2 - 64e^{3t}.$$

Compare the coefficients of the two sides:

$$\begin{aligned} -10a_0 - 7a_1 + 8a_2 &= 0, \\ -10a_1 - 14a_2 &= 0, \\ -10a_2 &= 100, \\ 32b &= -64. \end{aligned}$$

Solving this linear system we obtain

$$a_0 = -89/5, a_1 = 14, a_2 = -10, b = -2,$$

and thus

$$y_p = -\frac{89}{5} + 14t - 10t^2 - 2e^{3t}.$$

The general solutions to the nonhomogeneous equation are

$$y = y_p + y_c = -\frac{89}{5} + 14t - 10t^2 - 2e^{3t} + C_1e^{-t} + C_2e^{2t} + C_3e^{-5t},$$

where C_1, C_2, C_3 are free parameters.

(b) Examine the initial conditions:

$$\begin{aligned} C_1 + C_2 + C_3 - \frac{89}{5} - 2 &= -20, \\ -C_1 + 2C_2 - 5C_3 + 14 - 6 &= -\frac{29}{5}, \\ C_1 + 4C_2 + 25C_3 - 20 - 18 &= \frac{19}{5}. \end{aligned}$$

Solve this for C_1, C_2, C_3:

$$C_1 = -\frac{1}{5}, C_2 = -2, C_3 = 2.$$

Therefore, the solution of the initial value problem is

$$y = -\frac{89}{5} + 14t - 10t^2 - 2e^{3t} - \frac{1}{5}e^{-t} - 2e^{2t} + 2e^{-5t}.$$

Example 5: Solve $y^{(4)} + 8y'' + 16y = 64t\sin(2t)$.

Solution: First, solve the corresponding homogeneous equation

$$y_c^{(4)} + 8y_c'' + 16y_c = 0.$$

This was done in Example 2. The complementary solutions are

$$y_c = C_1\cos(2t) + C_2\sin(2t) + C_3t\cos(2t) + C_4t\sin(2t),$$

where C_1, C_2, C_3, C_4 are free parameters.

Next, set up a trial form for a particular solution:

$$y_p = t^2(a_0 + a_1t)\cos(2t) + t^2(b_0 + b_1t)\sin(2t).$$

Notice that this is the resonant case and the correcting term t^2 is needed. Plug y_p in the nonhomogeneous equation and simplify:

$$(-32a_0 + 48b_1 - 96a_1t)\cos(2t) + (-48a_1 - 32b_0 - 96b_1t)\sin(2t) = 64t\sin(2t).$$

Compare the coefficients of the two sides:

$$\begin{aligned}
&-32a_0 + 48b_1 = 0,\\
&-96a_1 = 0,\\
&-48a_1 - 32b_0 = 0,\\
&-96b_1 = 64.
\end{aligned}$$

Solving this we obtain:

$$a_0 = -1, a_1 = 0, b_0 = 0, b_1 = -2/3.$$

Therefore, the general solutions of the nonhomogeneous equation are

$$y = y_p + y_c = -t^2\cos(2t) - \frac{2}{3}t^3\sin(2t) + C_1\cos(2t) + C_2\sin(2t) + C_3t\cos(2t) + C_4t\sin(2t),$$

where C_1, C_2, C_3, C_4 are free parameters.

Example 6: Solve $y^{(6)} + y^{(5)} - 3y^{(4)} - 5y^{(3)} - 2y^{(2)} = -4 + 8t^2 - 27te^{-t} + 16te^t$.

Solution: First, solve the corresponding homogeneous equation

$$y_c^{(6)} + y_c^{(5)} - 3y_c^{(4)} - 5y_c^{(3)} - 2y_c^{(2)} = 0.$$

Find the roots of the characteristic polynomial:

$$\lambda^6 + \lambda^5 - 3\lambda^4 - 5\lambda^3 - 2\lambda^2 = \lambda^2(\lambda+1)^3(\lambda-2).$$

Hence, the complementary solutions are

$$y_c = C_1 + C_2 t + C_3 e^{-t} + C_4 te^{-t} + C_5 t^2 e^{-t} + C_6 e^{2t},$$

where $C_1, \cdots, C_6$ are free parameters.

Next, set up a trial form for a particular solution:

$$y_p = t^2(a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t)e^t + t^3(c_0 + c_1 t)e^{-t}.$$

Notice that this is a resonant case and we needed the red terms to modify y_p. Plugging y_p in the nonhomogeneous equation and comparing the coefficients of the two sides, we can determine the coefficients:

$$a_0 = -18, a_1 = 10/3, a_2 = -1/3, \quad b_0 = 5, b_1 = -2, \quad c_0 = 7/2, c_1 = 3/8.$$

Therefore, the general solutions of the nonhomogeneous equation are

$$\begin{aligned} y &= y_p + y_c \\ &= t^2\left(-18 + \frac{10}{3}t - \frac{1}{3}t^2\right) \quad + (5-2t)\,e^t \quad + t^3\left(\frac{7}{2} + \frac{3}{8}t\right)e^{-t} \\ &\quad + C_1 + C_2 t + C_3 e^{-t} + C_4 te^{-t} + C_5 t^2 e^{-t} + C_6 e^{2t}, \end{aligned}$$

where $C_1, \cdots, C_6$ are free parameters.

EXERCISES

Solve the following differential equations and initial value problems.

[1] (a) $y''' + 4y'' - y' - 4y = 0$ (b) $y''' + 4y'' - 3y' - 18y = 0$ (c) $y''' + 6y'' + 12y' + 8y = 0$

[2] $y''' - 5y'' - y' + 5y = 10t - 63e^{-2t} + 29\sin(2t)$.

[3] $y^{(4)} + 2y''' - 2y' - y = 24te^{-t} + 24e^{t} - 8\sin t, \quad y(0) = -2,\ y'(0) = 0,\ y''(0) = 6,\ y'''(0) = 10.$

[4] $y^{(4)} + 18y'' + 81y = 64t\cos t + 108t\cos(3t)$

Answers:

[1] (a) $y = C_1e^{-t} + C_2e^{t} + C_3e^{-4t}$ (b) $y = C_1e^{2t} + C_2e^{-3t} + C_3te^{-3t}$
(c) $y = C_1e^{-2t} + C_2te^{-2t} + C_3t^2e^{-2t}$

[2] $y = \dfrac{2}{5} + 2t + 3e^{-2t} + \dfrac{2}{5}\cos(2t) + \sin(2t) + C_1e^{-t} + C_2e^{t} + C_3e^{5t}$

[3] $y = t^3\left(-1 - \dfrac{1}{2}t\right)e^{-t} + 3te^{t} - 2\cos t - e^{t} + e^{-t} - te^{-t} - 2t^2e^{-t}$

[4] $y = t\cos t + \dfrac{1}{2}\sin t - \dfrac{1}{2}t^3\cos(3t) + \dfrac{1}{2}t^2\sin(3t) + C_1\cos(3t) + C_2\sin(3t) + C_3t\cos(3t) + C_4t\sin(3t)$