PROBLEMS

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 150 liter (L) of a dye solution with a concentration of 3 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 3 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 2% of its original value.

2. A tank initially contains 200 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 4 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. A tank originally contains 160 gal of fresh water. Then water containing $\frac{1}{4}$ lb of salt per gallon is poured into the tank at a rate of 4 gal/min, and the mixture is allowed to leave at the same rate. After 8 min the process is stopped, and fresh water is poured into the tank at a rate of 6 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 8 min.

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

5. A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

(a) Find the amount of salt in the tank at any time.

(b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.

(c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

6. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h.

(a) Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

(b) By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}, \qquad \text{(i)}$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. The value of α for water is about 0.6.

(c) Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

7. An outdoor swimming pool loses 0.05% of its water volume every day it is in use, due to losses from evaporation and from excited swimmers who splash water. A system is available to continually replace water at a rate of G gallons per day of use.

(a) Find an expression, in terms of G, for the equilibrium volume of the pool. Sketch a few graphs for the volume $V(t)$, including all possible types of solutions.

(b) If the pool volume is initially 1% above its equilibrium value, find an expression for $V(t)$.

(c) What is the replacement rate G required to maintain 12,000 gal of water in the pool?

8. Cholesterol is produced by the body for the construction of cell walls, and is also absorbed from certain foods. The blood cholesterol level is measured in units of milligrams per deciliter, or mg/dl. The net cholesterol production or destruction by the body is modeled by a rate r per day, times the difference between the body's "natural" cholesterol level (a constant) and the actual cholesterol level at any time t. The rate of absorption from food is estimated as a constant k in milligrams per deciliter per day.

(a) A person's cholesterol level at the start of a testing period is 150 mg/dl. Find an expression for the cholesterol level at any subsequent time t. If the rate r is 0.10 per day and the natural level is 100 mg/dl, find the cholesterol level of the person 10 days after the start of the testing period, in terms of k.

(b) If $k = 25$, what is the cholesterol level of this person after a long time?

(c) Suppose this person starts a low-cholesterol diet. What must the value of k be so that the long-time cholesterol level is 180 mg/dl?

9. Imagine a medieval world. In this world a Queen wants to poison a King, who has a wine keg with 500 L of his favorite wine. The Queen gives a conspirator a liquid containing 5 g/L of poison, which must be poured slowly into the keg at a rate

TABLE 2.3.1 Volume and flow data for the Great Lakes.

Lake	V (km^3 × 10^3)	r (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

22. A ball with mass 0.25 kg is thrown upward with initial velocity 24 m/s from the roof of a building 26 m high. Neglect air resistance.

(a) Find the maximum height above the ground that the ball reaches.
(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.
(c) Plot the graphs of velocity and position versus time.

23. Assume that conditions are as in Problem 22 except that there is a force due to air resistance of $|v|/30$, where the velocity v is measured in meters per second.

(a) Find the maximum height above the ground that the ball reaches.
(b) Find the time that the ball hits the ground.
(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 22.

24. Assume that conditions are as in Problem 22 except that there is a force due to air resistance of $v_2/1325$, where the velocity v_2 is measured in meters per second.

(a) Find the maximum height above the ground that the ball reaches.
(b) Find the time that the ball hits the ground.
(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 22 and 23.

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

26. A rocket sled having an initial speed of 160 mi/h is slowed by a channel of water. Assume that, during the braking process, the acceleration a is given by $a(v) = -\mu v^2$, where v is the velocity and μ is a constant.

(a) As in Example 4 in the text, use the relation $dv/dt = v(dv/dx)$ to write the equation of motion in terms of v and x.
(b) If it requires a distance of 2200 ft to slow the sled to 16 mi/h, determine the value of μ.
(c) Find the time τ required to slow the sled to 16 mi/h.

27. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.

(a) Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.
(b) Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right],$$

$$x_m = \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right].$$

(c) Show that the quantity kv_0/mg is dimensionless.

28. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

(a) Find the velocity $v(t)$ of the body at any time.
(b) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$, that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?
(c) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$, that is, as the mass approaches zero.

29. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law, $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.

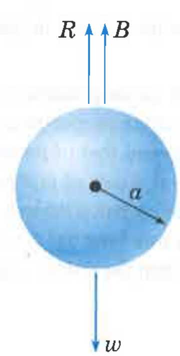
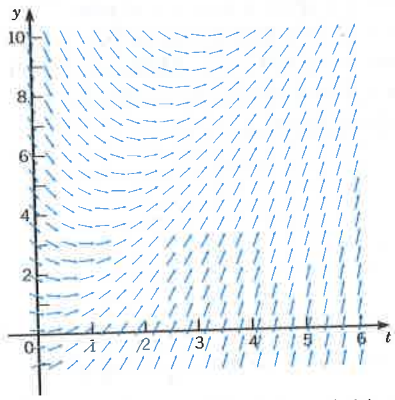


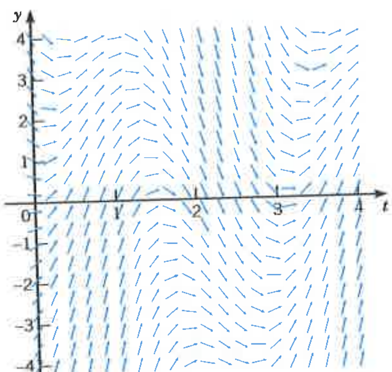
FIGURE 2.3.5 A body falling in a dense fluid.

9. (a)



(b) all solutions increase without bound (c) $y = ce^{-t/2} + 3t - 6$

11. (a)



(b) all solutions converge to an oscillatory function (c) $y = ce^{-t} - 2\cos 2t + \sin 2t$

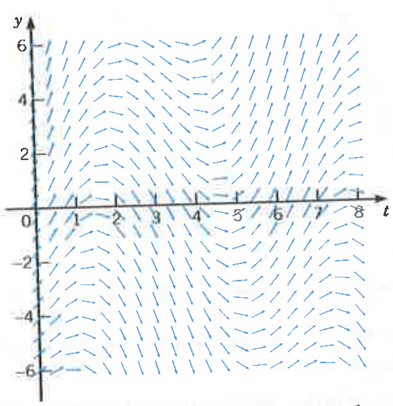
13. $y = 3e^t - 2e^{2t} + 2te^{2t}$

15. $y = 1/4 + t^2/6 - t/5 + t^{-4}/30$

17. $y = 2e^{2t} + te^{2t}$

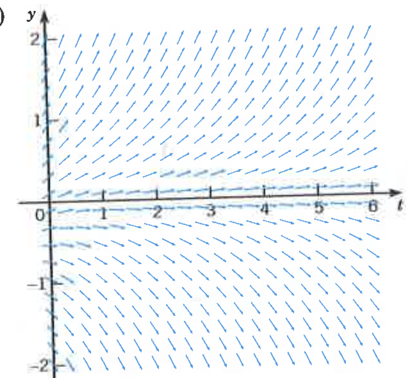
19. $y = -e^{-t}/t^4 - e^{-t}/t^3$

21. (a)



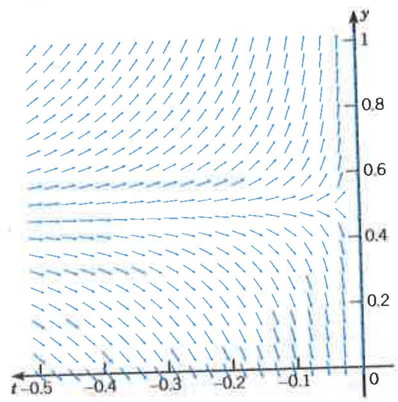
solutions increase or decrease without bound, $a_0 \approx -1$
(b) $y = (a + 9/10)e^{t/3} - (9/10)\cos t + (27/10)\sin t$
(c) $a_0 = -9/10$

23. (a)



solutions increase or decrease without bound, $a_0 \approx 0$
(b) $y = [(2 + a(3\pi + 4))e^{2t/3} - 2e^{-\pi t/2}]/(3\pi + 4)$ (c) $a_0 = -2/(3\pi + 4)$

25. (a)



solutions increase or decrease without bound, $a_0 \approx 0.5$
(b) $y = t^{-2}(a\pi^2/4 - \cos t)$ (c) $a_0 = 4/\pi^2$

27. (1.3643, 0.8201) approximately

29. (a) $y = ce^{-t/4} + 12 + (8\cos 2t + 64\sin 2t)/65$, solution converges toward an oscillating function
(b) approximately 10.066

31. $y_0 = -16/3$, solution decreases without bound

35. $y' + y = 3 - t$

37. $y' + y = 2 - 2t - t^2$

41. $y = ct^{-1} + (3/4)t^{-1}\cos 2t + (3/2)\sin 2t$

43. $y = ce^{-t/2} + 3t^2 - 12t + 24$

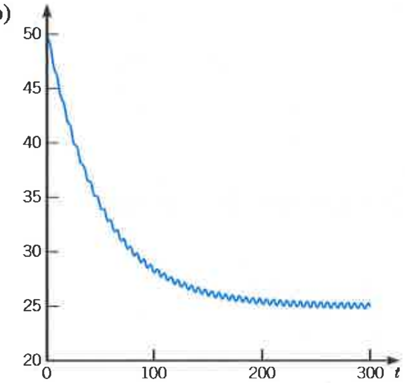
Section 2.3 Modeling with First Order Equations *page 65*

1. $t = 50 \ln 50 \approx 195.6$ min

3. $Q = 40(e^{-3/10} - e^{-1/2}) \approx 5.37$ lb

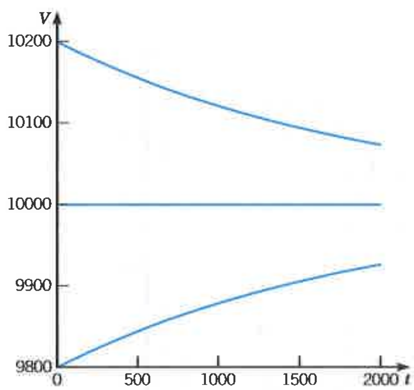
5. (a) $Q(t) = 25 - (625/2501)\cos t + (25/5002)\sin t + (63150/2501)e^{-t/50}$

(b)



(c) level = 25, amplitude = $25\sqrt{2501}/5002 \approx 0.24995$

7. (a) $V = 2000G$, graph depicts $G = 5$



(b) $V(t) = 2000G + 20Ge^{-t/2000}$ (c) $G = 6$ gal

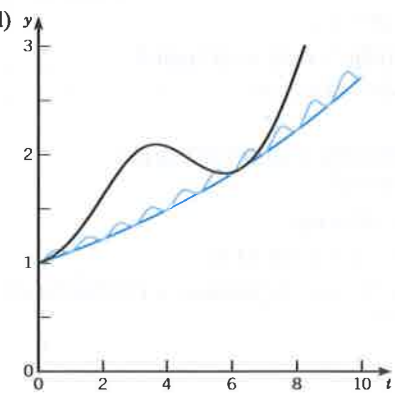
9. (a) $Q(t) = 2500(1 - e^{-t/1000})$ m
(b) $t = 1000\ln(1000/999) \approx 1$ min (c) 1 min

11. (a) $S(t) = k(e^{rt} - 1)/r$ (b) $k \approx \$6,060.99$ (c) 6.92%

13. (a) $t \approx 135.36$ months (b) \$152, 698.56

15. (a) $1.2097 \cdot 10^{-4}$ year^{-1} (b) $Q(t) = Q_0 e^{-1.2097t/10^4}$
(c) 5, 730 years

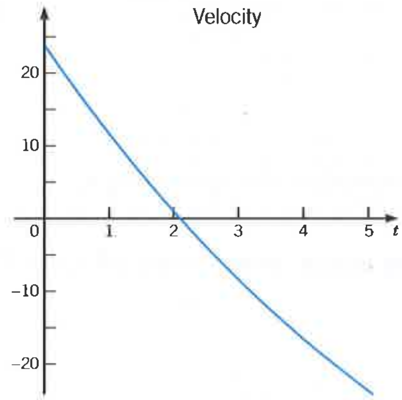
17. (a) $\tau \approx 2.9632$; no (b) $\tau = 10\ln 2 \approx 6.9315$
(c) $\tau \approx 6.3804$
(d)

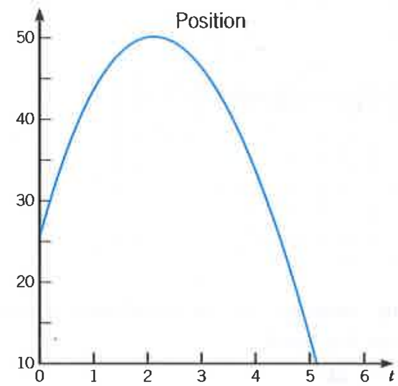


19. $t = \ln(13/8)/\ln(13/12) \approx 6.07$ min

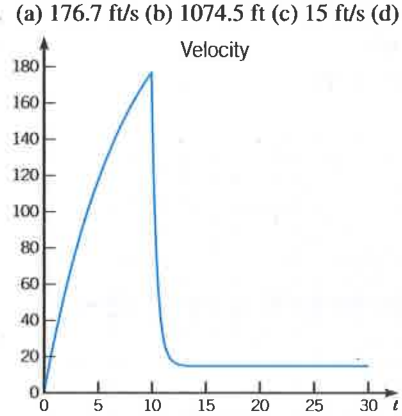
21. (a) $c(t) = k + (P/r) + (c_0 - k - (P/r))e^{-rt/V}$; $\lim_{t\to\infty} c(t) = k + (P/r)$ (b) $T = (V\ln 2)/r$, $T = (V\ln 10)/r$
(c) Superior: $T = 430.9$ years, Michigan: $T = 71.4$ years, Erie: $T = 6.1$ years, Ontario: $T = 17.6$ years

23. (a) 50.22 m (b) 5.56 s
(c)





25. (a) 176.7 ft/s (b) 1074.5 ft (c) 15 ft/s (d) 256.6 s
(e)



27. (a) $x_m = mv_0/k - (m^2g/k^2)\ln(1 + kv_0/mg)$; $t_m = (m/k)\ln(1 + kv_0/mg)$

29. (a) $v_L = 2a^2g(\rho' - \rho)/9\mu$ (b) $e = 4\pi a^3 g(\rho' - \rho)/3E$

31. (a) $v = R\sqrt{2g/(R+x)}$ (b) 50.6 hours

33. (a) $v = (u\cos A)e^{-rt}$, $w = -g/r + (u\sin A + g/r)e^{-rt}$
(b) $x = u\cos A(1 - e^{-rt})/r$, $y = -gt/r + (u\sin A + g/r)(1 - e^{-rt})/r + h$