For many applied mathematicians, engineers, and scientists, mathematical modeling is akin to poetry—an art form and creative act employing language that adheres to form and conventions. Likewise, there are rules (e.g., physical laws) that the mathematical modeler must follow, yet he or she has access to a myriad of mathematical tools (the language) for describing the phenomenon under investigation. History abounds with the names of scientists, mathematicians, and engineers, driven by the desire to understand nature and advance technology, who have engaged in the practice of mathematical modeling: Newton, Euler, von Kármán, Verhulst, Maxwell, Rayleigh, Navier, Stokes, Heaviside, Einstein, Schrödinger, and so on. Their contributions have literally changed the world. Nowadays, mathematical modeling is carried out in universities, government agencies and laboratories, business and industrial concerns, policy think tanks, and institutes dedicated to research and education. For many practitioners of mathematical modeling, it is, in a sense, their *raison d'être*.

PROBLEMS

1. Newton's Law of Cooling. A cup of hot coffee has a temperature of 200°F when freshly poured, and is left in a room at 70°F. One minute later the coffee has cooled to 190°F.

(a) Assume that Newton's law of cooling applies. Write down an initial value problem that models the temperature of the coffee.

(b) Determine when the coffee reaches a temperature of 170°F.

2. Blood plasma is stored at 40°F. Before it can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 minutes (min) for the plasma to warm to 90°F. Assume Newton's law of cooling applies. How long will it take the plasma to warm to 90°F if the oven temperature is set at 100°F?

3. At 11:09 p.m. a forensics expert arrives at a crime scene where a dead body has just been found. Immediately, she takes the temperature of the body and finds it to be 80°F. She also notes that the programmable thermostat shows that the room has been kept at a constant 68°F for the past 3 days. After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5°F. This last temperature reading was taken exactly one hour after the first one. The next day the investigating detective asks the forensic expert, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°F) prior to death, what does she tell the detective?

4. Population Problems. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

(a) Find the rate constant r if the population doubles in 30 days.

(b) Find r if the population doubles in N days.

5. The field mouse population in Example 3 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.

(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

(c) Find the initial population p_0 if the population is to become extinct in 1 year.

6. Radioactive Decay. Experiments show that a radioisotope decays at a rate negatively proportional to the amount of the isotope present.

(a) Use the following variables and parameters to write down and solve an initial value problem for the process of radioactive decay: t = time; $a(t)$ = amount of the radioisotope present at time t; a_0 = initial amount of radioisotope; r = decay rate, where $r > 0$.

(b) The **half-life**, $T_{1/2}$, of a radioisotope is the amount of time it takes for a quantity of the radioactive material to decay to one-half of its original amount. Find an expression for $T_{1/2}$ in terms of the decay rate r.

7. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 milligrams (mg) of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

(b) Find an expression for the amount of thorium-234 present at any time t.

(c) Find the time required for the thorium-234 to decay to one-half its original amount.

8. Classical Mechanics. The differential equation for the velocity v of an object of mass m, restricted to vertical motion and subject only to the forces of gravity and air resistance, is

$$m\frac{dv}{dt} = -mg - \gamma v. \qquad \text{(i)}$$

In Eq. (i) we assume that the drag force, $-\gamma v$ where $\gamma > 0$ is a drag coefficient, is proportional to the velocity.

Acceleration due to gravity is denoted by g. Assume that the upward direction is positive.

(a) Show that the solution of Eq. (i) subject to the initial condition $v(0) = v_0$ is

$$v = \left(v_0 + \frac{mg}{\gamma}\right)e^{-\gamma t/m} - \frac{mg}{\gamma}.$$

(b) Sketch some integral curves, including the equilibrium solution, for Eq. (i). Explain the physical significance of the equilibrium solution.

(c) If a ball is initially thrown in the upward direction so that $v_0 > 0$, show that it reaches its maximum height when

$$t = t_{\text{max}} = \frac{m}{\gamma}\ln\left(1 + \frac{\gamma v_0}{mg}\right).$$

(d) The terminal velocity of a baseball dropped from a high tower is measured to be 33 m/s. If the mass of the baseball is 145 grams (g) and $g = 9.8$ m/s^2, what is the value of γ?

(e) Using the values for m, g, and γ in part (d), what would be the maximum height attained for a baseball thrown upward with an initial velocity $v_0 = 30$ m/s from a height of 2 m above the ground?

9. For small, slowly falling objects, the assumption made in Eq. (i) of Problem 8 that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

(a) Write a differential equation for the velocity of a falling object of mass m if the drag force is proportional to the square of the velocity. Assume that the upward direction is positive.

(b) Determine the limiting velocity after a long time.

(c) If $m = 0.025$ kilograms (kg), find the drag coefficient so that the limiting velocity is -35 m/s.

Mixing Problems. Many physical systems can be cast in the form of a mixing tank problem. Consider a tank containing a solution—a mixture of solute and solvent–such as salt dissolved in water. Assume that the solution at concentration $c_i(t)$ flows into the tank at a volume flow rate $r_i(t)$ and is simultaneously pumped out at the volume flow rate $r_o(t)$. If the solution in the tank is well mixed, then the concentration of the outflow is $Q(t)/V(t)$, where $Q(t)$ is the amount of solute at time t and $V(t)$ is the volume of solution in the tank. The differential equation that models the changing amount of solute in the tank is based on the principle of conservation of mass,

$$\underbrace{\frac{dQ}{dt}}_{\text{rate of change of } Q(t)} = \underbrace{c_i(t)r_i(t)}_{\text{rate in}} - \underbrace{\{Q(t)/V(t)\}\, r_o(t)}_{\text{rate out}}, \qquad \text{(i)}$$

where $V(t)$ also satisfies a mass conservation equation,

$$\frac{dV}{dt} = r_i(t) - r_o(t). \qquad \text{(ii)}$$

If the tank initially contains an amount of solute Q_0 in a volume of solution, V_0, then initial conditions for Eqs. (i) and (ii) are $Q(0) = Q_0$ and $V(0) = V_0$, respectively.

10. A tank initially contains 200 liters (L) of pure water. A solution containing 1 g/L enters the tank at a rate of 4 L/min, and the well-stirred solution leaves the tank at a rate of 5 L/min. Write initial value problems for the amount of salt in the tank and the amount of brine in the tank, at any time t.

11. A tank contains 100 gallons (gal) of water and 50 ounces (oz) of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture flows out at the same rate. Write an initial value problem for the amount of salt in the tank at any time t.

12. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.

(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

13. Pharmacokinetics. A simple model for the concentration $C(t)$ of a drug administered to a patient is based on the assumption that the rate of decrease of $C(t)$ is negatively proportional to the amount present in the system,

$$\frac{dC}{dt} = -kC,$$

where k is a rate constant that depends on the drug and its value can be found experimentally.

(a) Suppose that a dose administered at time $t = 0$ is rapidly distributed throughout the body, resulting in an initial concentration C_0 of the drug in the patient. Find $C(t)$, assuming the initial condition $C(0) = C_0$.

(b) Consider the case where doses of C_0 of the drug are given at equal time intervals T, that is, doses of C_0 are administered at times $t = 0, T, 2T, \ldots$. Denote by C_n the concentration immediately after the nth dose. Find an expression for the concentration C_2 immediately after the second dose.

(c) Find an expression for the concentration C_n immediately after the nth dose. What is $\lim_{n\to\infty} C_n$?

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106*, no. 2 (1999), pp. 127–135.

14. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time?

Continuously Compounded Interest. The amount of money $P(t)$ in an interest bearing account in which the principal is compounded continuously at a rate r per annum and in which money is continuously added, or subtracted, at a rate of k dollars per annum satisfies the differential equation

$$\frac{dP}{dt} = rP + k. \tag{i}$$

The case $k < 0$ corresponds to paying off a loan, while $k > 0$ corresponds to accumulating wealth by the process of regular contributions to an interest bearing savings account.

15. Show that the solution to Eq. (i), subject to the initial condition $P(0) = P_0$, is

$$P = \left(P_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r}. \tag{ii}$$

Use Eq. (ii) in Problem 15 to solve Problems 16 and 17.

16. According to the International Institute of Social History (Amsterdam), the amount of money used to purchase Manhattan Island in 1626 is valued at \$1,050 in terms of today's dollars. If that amount were instead invested in an account that pays 4% per annum with continuous compounding, what would be the value of the investment in 2020? Compare with the case that interest is paid at 6% per annum.

17. How long will it take to pay off a student loan of \$20,000 if the interest paid on the principal is 5% and the student pays \$200 per month. What is the total amount of money repaid by the student?

18. Derive Eq. (ii) in Problem 15 from the discrete approximation to the change in the principal that occurs during the time interval $[t, t + \Delta t]$,

$$P(t + \Delta t) \cong P(t) + (r\Delta t)P(t) + k\Delta t,$$

assuming that $P(t)$ is continuously differentiable on $t \geq 0$. [*Hint:* Substitute $P(t + \Delta t) = P(t) + P'(t)\Delta t + (1/2)P''(\hat{t})(\Delta t)^2$), where $t < \hat{t} < t + \Delta t$, simplify, divide by Δt, and let $\Delta t \to 0$.]

Miscellaneous Modeling Problems

19. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

20. Archimedes's *principle of buoyancy* states that an object submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced. An experimental, spherically shaped sonobuoy of radius 1/2 m with a mass m kg is dropped into the ocean with a velocity of 10 m/s when it hits the water. The sonobuoy experiences a drag force due to the water equal to one-half its velocity. Write down a differential equation describing the motion of the sonobuoy. Find values of m for which the sonobuoy will sink and calculate the corresponding terminal sink velocity of the sonobuoy. The density of seawater is $\rho_0 = 1.025$ kg/L.

1.2 Qualitative Methods: Phase Lines and Direction Fields

In Section 1.1 we were able to find solutions of the differential equations

$$\frac{du}{dt} = -k(u - T_0) \qquad \text{and} \qquad \frac{dp}{dt} = rp - k \tag{1}$$

by using a simple integration technique. Do not assume that this is always possible. Finding closed-form analytic solutions of differential equations can be difficult or impossible. Fortunately, it is possible to obtain information about the qualitative behavior of solutions by using elementary ideas from calculus and graphical methods; we consider two such methods in this section—phase line diagrams and direction fields.

Qualitative behavior refers to general properties of the differential equation and its solutions such as existence of equilibrium points, behavior of solutions near equilibrium