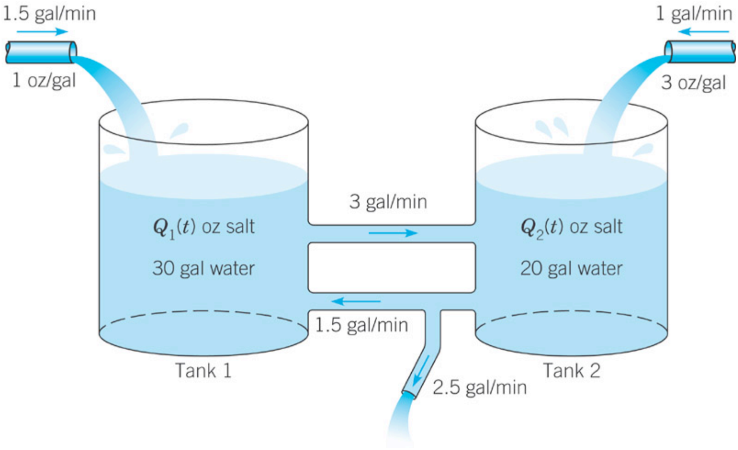
Some Applications of 2-D Systems of Differential Equations

Examples:

- Salt in Tanks (a linear system)

- Population Model - Competing Species (a nonlinear system)

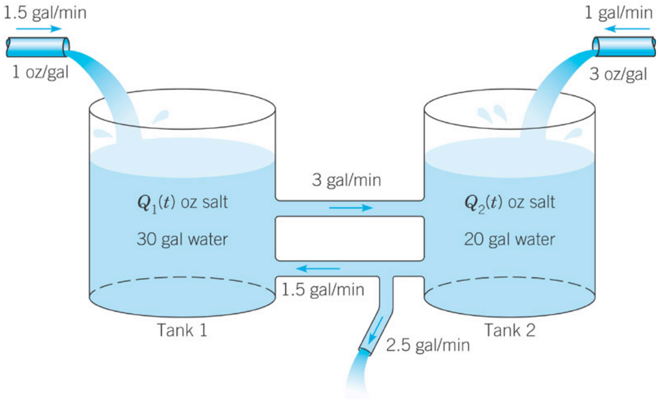
Example 1. Salt in Tanks (a linear system)



Question: Given initial condition $Q_1(0) = 55$ oz, $Q_2(0) = 26$ oz,

find $Q_1(t), Q_2(t)$.

Example 1. (continued. Set up equations.)



Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) – (rate of salt out)

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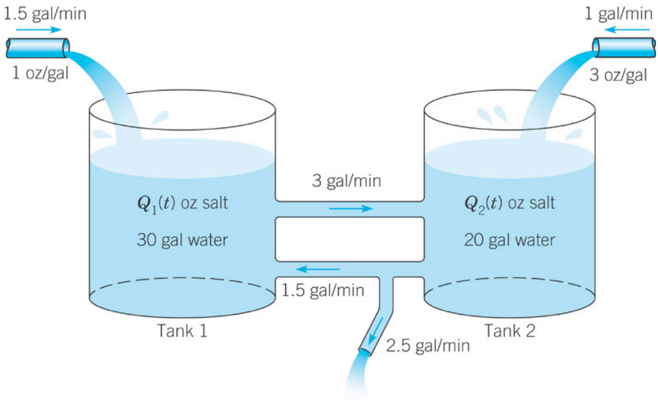

Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) − (rate of salt out)

Tank 1: $$Q_1'(t) = 1.5 \times 1 \quad + 1.5 \times \frac{Q_2(t)}{20} \quad - 3 \times \frac{Q_1(t)}{30} \quad (\text{oz/min})$$

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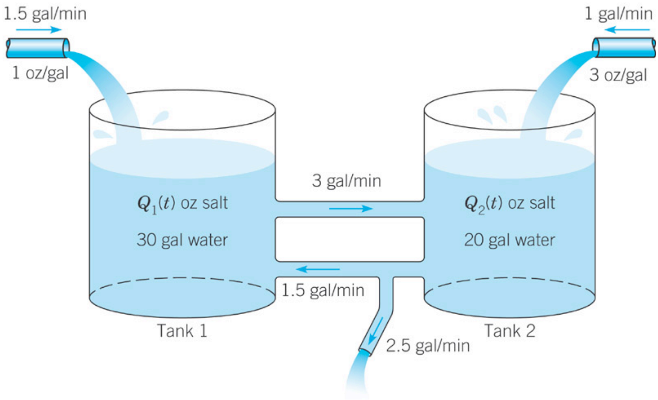

Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) – (rate of salt out)

Tank 2: $$Q_2'(t) = 1 \times 3 + 3 \times \frac{Q_1(t)}{30} - (1.5 + 2.5) \times \frac{Q_2(t)}{20} \quad (\text{oz/min})$$

Example 1. (continued)

The System of Diff Eqs

$$\begin{cases} Q_1' &= -0.1\ Q_1 + 0.075\ Q_2 + 1.5 \\ Q_2' &= 0.1\ Q_1 - 0.2\ Q_2 + 3 \end{cases}$$

The Matrix Form

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

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System Recasted

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 - 42 \\ Q_2 - 36 \end{bmatrix}$$

Example 1. (continued)

2-D System

$$\vec{\mathbf{Q}}' = A\left(\vec{\mathbf{Q}} - \vec{\mathbf{a}}\right)$$

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The coefficient matrix

$$A = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix}$$

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Eigenvalues & Eigenvectors of A:

$$\lambda_1 = -0.05, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}; \qquad \lambda_2 = -0.25, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

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General Solutions:

$$\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + C_1 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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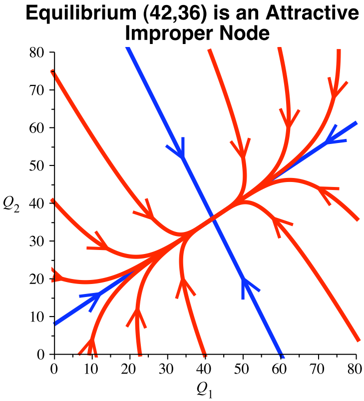

Example 1. (continued. Initial Value Problem)

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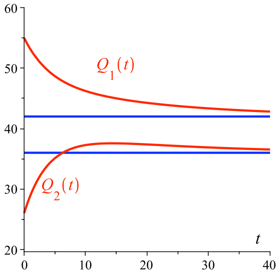


Solution: $\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + 2e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 7e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Examples:

- Salt in Tanks (a linear system)
- Population Model - Competing Species (a nonlinear system)

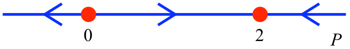
$$P' = rP\left(1 - \frac{P}{K}\right)$$

where r is the net per capita growth rate when $P \approx 0$,
K is the *carrying capacity*.

Solution Formula: $P(t) = \dfrac{KP(0)}{P(0) + [K - P(0)]e^{-rt}}$

Example. $P' = 6P(1 - P/2)$ $(r = 6,\ K = 2)$

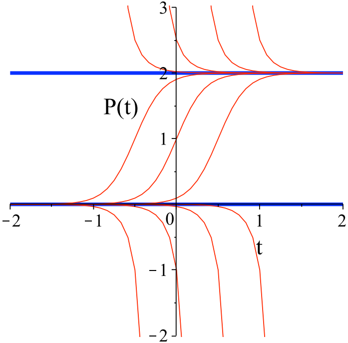
Phase portrait



Equilibrium $P = 0$ is unstable.

Equilibrium $P = K$ is asymp stable.

All positive solutions $P(t)$ converge to K as $t \to \infty$.



Logistic Dynamics of Two Species

If no interactions: $\begin{cases} x_1' = x_1(6 - 3x_1) \\ x_2' = x_2(3 - 2x_2) \end{cases}$

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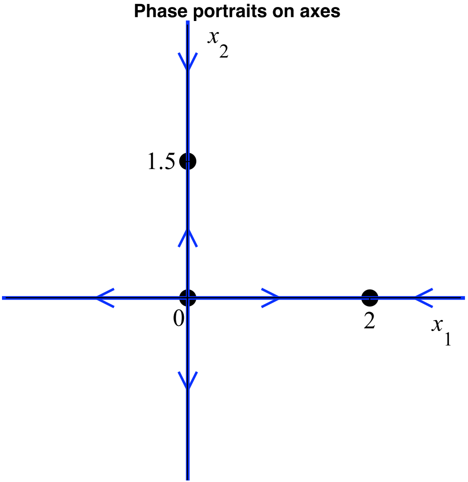

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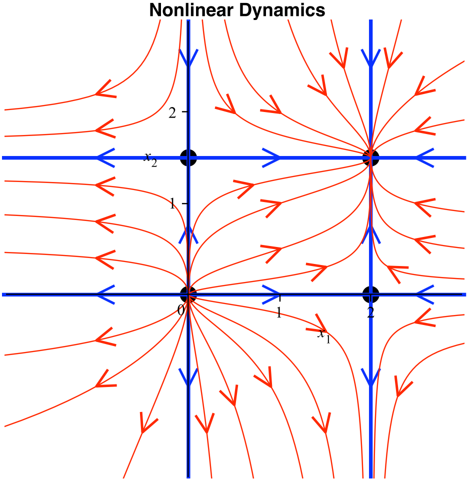

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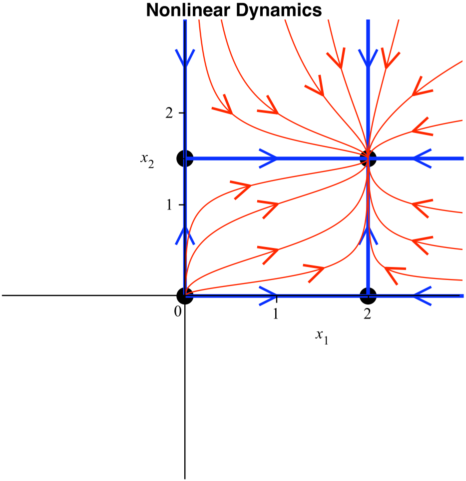

Example 3. Logistic Growth & Competition

Lotka (1925), Volterra (1926), Gause (1934), $\cdots\cdots$

With Competition:

$$\begin{cases} x_1' = x_1(6 - 3x_1 \; \boxed{-2x_2} \;) \\ x_2' = x_2(3 - 2x_2 \; \boxed{-x_1} \;) \end{cases}$$

x_2 reduces the growth of x_1

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Basic Questions:

- Find equilibria. (i.e., time independent solutions)
- Construct a linear approximating system near each equilibrium.
  (use the Jacobian matrix, that is, partial derivatives)
- Study the linear approximating dynamics near the equilibrium.
  (use eigenvalues & eigenvectors)
- Determine the nonlinear dynamics near the equilibrium.
  (if eigenvalues are $\neq 0$ & are not purely imaginary, Yes We Can!)

Example 3. (Continued. Find equilibria.)

Competing Species:

$$\begin{cases} x_1' = x_1(6 - 3x_1 - 2x_2) \\ x_2' = x_2(3 - x_1 - 2x_2) \end{cases}$$

$$\begin{aligned} f_1(x_1, x_2) &= x_1(6 - 3x_1 - 2x_2) \\ f_2(x_1, x_2) &= x_2(3 - x_1 - 2x_2) \end{aligned}$$

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Equilibria:

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$$\begin{cases} x_1(6 - 3x_1 - 2x_2) = 0 \\ x_2(3 - x_1 - 2x_2) = 0 \end{cases} \Longrightarrow \text{Separate to four combinations}$$

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Four equilibria:

$$(x_1, x_2) = (0, 0), \quad (2, 0), \quad (0, \tfrac{3}{2}), \quad (\tfrac{3}{2}, \tfrac{3}{4}).$$

Example 3. (continued. Linear Approximation)

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Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$**:**

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Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$**:**

- Prepare the Jacobian matrix:

$$J = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 6 - 6x_1 - 2x_2 & -2x_1 \\ -x_2 & 3 - x_1 - 4x_2 \end{bmatrix}$$

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- Evaluate J at equilibrium $(x_1, x_2) = (\frac{3}{2}, \frac{3}{4})$:

$$J = \begin{bmatrix} -\frac{9}{2} & -3 \\ -\frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

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$$\begin{cases} x_1' = x_1(6 - 3x_1 - 2x_2) \\ x_2' = x_2(3 - x_1 - 2x_2) \end{cases} \qquad \begin{vmatrix} f_1(x_1, x_2) = x_1(6 - 3x_1 - 2x_2) \\ f_2(x_1, x_2) = x_2(3 - x_1 - 2x_2) \end{vmatrix}$$

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Example 3. Linear dynamics near $(0, 0)$

The Linear Approximating System near equilibrium $(0, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Eigenvalues & Eigenvectors:

$$\lambda_1 = 6, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 3. Linear dynamics near $(0, 0)$

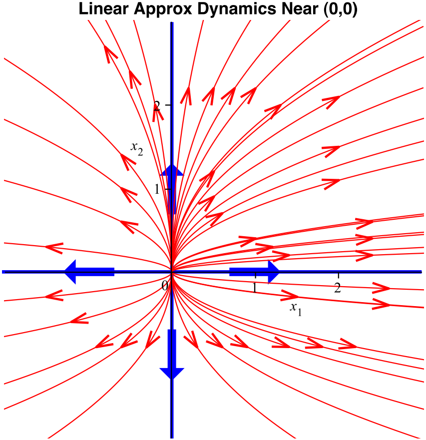
The Linear Approximating System near equilibrium $(0, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = 6, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Equilibrium $(0, 0)$ is a repulsive improper node.

Example 3. Linear dynamics near $(2, 0)$

The Linear Approximating System near equilibrium $(2, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 \end{bmatrix}$$

Example 3. Linear dynamics near $(2, 0)$

The Linear Approximating System near equilibrium $(2, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = -6, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

Example 3. Linear dynamics near $(2,0)$

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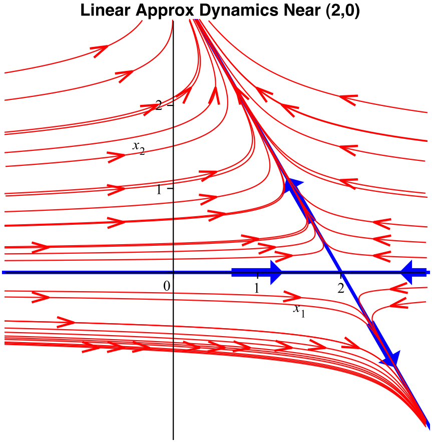

Equilibrium $(2,0)$ is a saddle

Example 3. Linear dynamics near $(0, \frac{3}{2})$

The Linear Approximating System near equilibrium $(0, \frac{3}{2})$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -\frac{3}{2} & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - \frac{3}{2} \end{bmatrix}$$

Example 3. Linear dynamics near $(0, \frac{3}{2})$

The Linear Approximating System near equilibrium $(0, \frac{3}{2})$:

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Eigenvalues & Eigenvectors:

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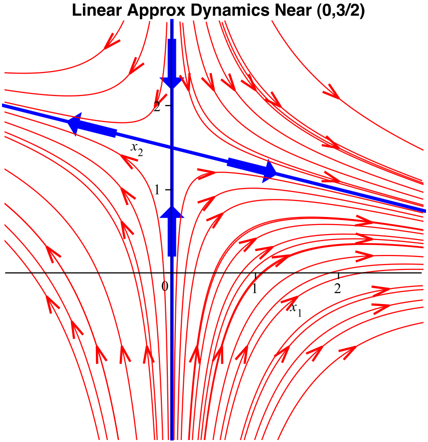

Equilibrium $(0, \frac{3}{2})$ is a saddle

Example 3. Linear dynamics near $(\frac{3}{2}, \frac{3}{4})$

The Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & -3 \\ -\frac{3}{4} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 - \frac{3}{2} \\ x_2 - \frac{3}{4} \end{bmatrix}$$

Example 3. Linear dynamics near $(\frac{3}{2}, \frac{3}{4})$

The Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$:

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Eigenvalues & Eigenvectors:

$$\lambda_1 = -3 + \tfrac{3}{2}\sqrt{2} \approx -0.88$$

$$\vec{\mathbf{w}}_1 = \begin{bmatrix} 2 \\ -1 - \sqrt{2} \end{bmatrix}$$

$$\lambda_2 = -3 - \tfrac{3}{2}\sqrt{2} \approx -5.12$$

$$\vec{\mathbf{w}}_2 = \begin{bmatrix} 2 \\ -1 + \sqrt{2} \end{bmatrix}$$

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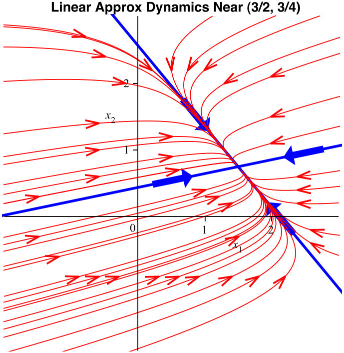
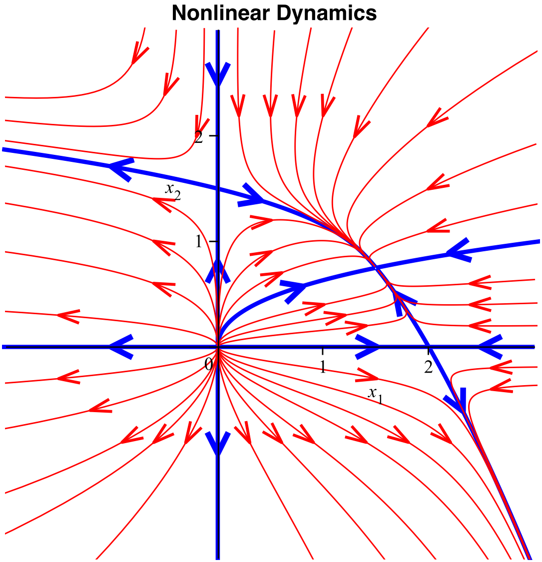
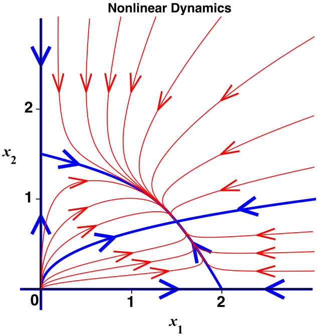


Equilibrium $(\frac{3}{2}, \frac{3}{4})$ is an attractive improper node.

Example 3. Global phase portrait

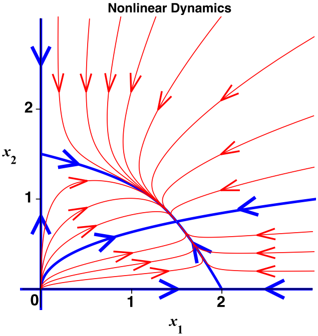


Example 3. Discussion



Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.



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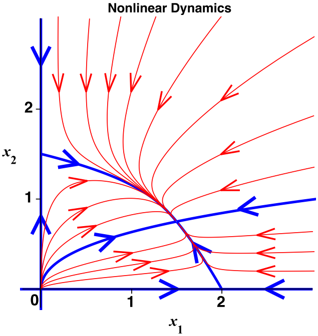
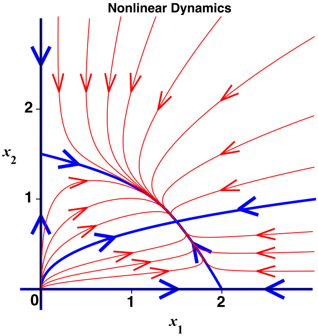
- The survival-extinction states $(2,0)$ and $(0,\frac{3}{2})$ are unstable.
- The co-existence state $(\frac{3}{2},\frac{3}{4})$ is asymptotically stable.



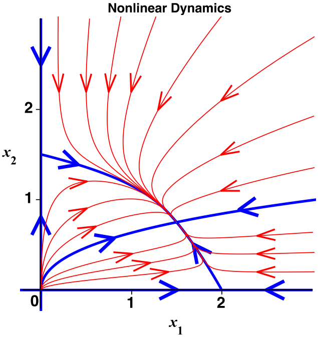
Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.
- The co-existence state $(\frac{3}{2}, \frac{3}{4})$ is asymptotically stable.
- All positive solutions converge to the co-existence state $(\frac{3}{2}, \frac{3}{4})$.



Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.
- The co-existence state $(\frac{3}{2}, \frac{3}{4})$ is asymptotically stable.
- All positive solutions converge to the co-existence state $(\frac{3}{2}, \frac{3}{4})$.
- A change of the initial populations does not affect the eventual convergence to the co-existence state $(\frac{3}{2}, \frac{3}{4})$.



Example 3. Discussion

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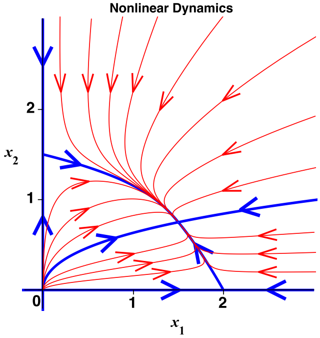

Question: Why is the co-existence stable in this system?

Example 3. Discussion

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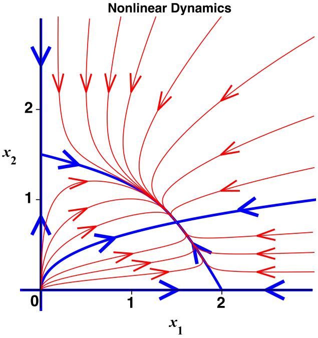


Question: Why is the co-existence stable in this system?

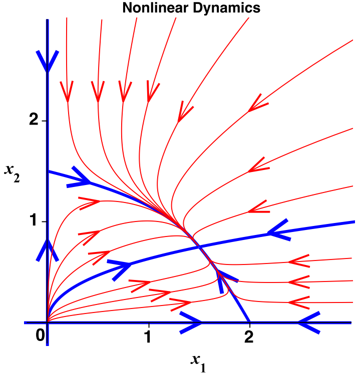
Answer: Weak competition.

Example 3. (continued. Weak competition)

$$\begin{cases} x_1' = x_1(\ 6 \ \ -3x_1 \ \ -2x_2 \ ) \\ x_2' = x_2(\ 3 \ \ -x_1 \ \ -2x_2 \ ) \end{cases}$$

The competition terms

$-2x_2$ and $-x_1$



Example 3. (continued. Weak competition)

$$\begin{cases} x_1' = x_1(\ 6\ \boxed{-3x_1}\ \boxed{-2x_2}\ ) \\ x_2' = x_2(\ 3\ \boxed{-x_1}\ \boxed{-2x_2}\ ) \end{cases}$$

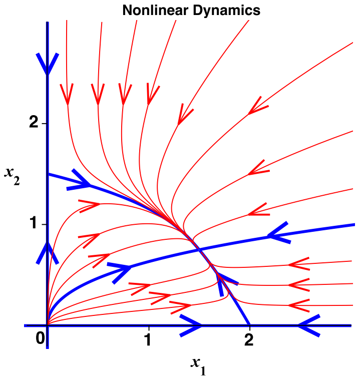
The competition terms

$\boxed{-2x_2}$ and $\boxed{-x_1}$

are "weaker" than

the resource inhibition terms

$\boxed{-3x_1}$ and $\boxed{-2x_2}$



Example 4. Strong Competition Model.

Competing Species:

$$\begin{cases} x_1' = x_1(3 - x_1 \; -2x_2) \\ x_2' = x_2(2 \; -x_1 \; - x_2) \end{cases}$$

x_2 reduces the growth of x_1

x_1 reduces the growth of x_2

Example 4. Strong Competition Model.

Competing Species:

$$\begin{cases} x_1' = x_1(3 - x_1 \ \boxed{-2x_2}\ ) \\ x_2' = x_2(2 \ \boxed{-x_1}\ - x_2\ ) \end{cases}$$

x_2 reduces the growth of x_1

x_1 reduces the growth of x_2

Equilibria:

$$\begin{cases} x_1(3 - x_1 - 2x_2) = 0 \\ x_2(2 - x_1 - x_2) = 0 \end{cases} \implies \text{Separate to four combinations}$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \qquad \begin{cases} 3 - x_1 - 2x_2 = 0 \\ x_2 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ 2 - x_1 - x_2 = 0 \end{cases} \qquad \begin{cases} 3 - x_1 - 2x_2 = 0 \\ 2 - x_1 - x_2 = 0 \end{cases}$$

Four equilibria: $(x_1, x_2) = (0,0), \quad (3,0), \quad (0,2), \quad (1,1).$

Example 4. Linear dynamics near $(0, 0)$

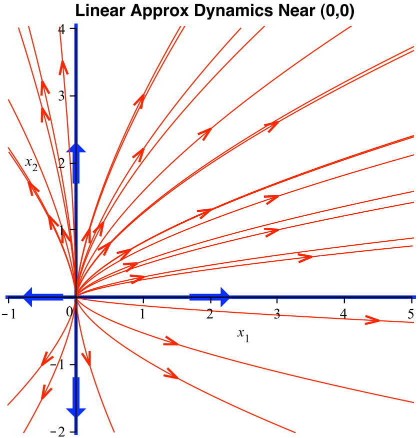
The Linear Approximating System near equilibrium $(0, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = 3, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Equilibrium $(0, 0)$ is a repulsive improper node.

Example 4. Linear dynamics near $(3, 0)$

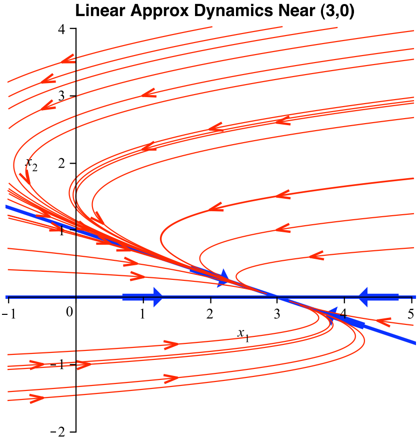
The Linear Approximating System near equilibrium $(3, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 3 \\ x_2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = -3, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$



Equilibrium $(3, 0)$ is
an attractive improper node

Example 4. Linear dynamics near $(0, 2)$

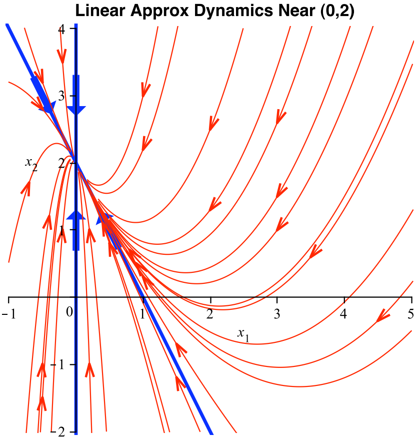
The Linear Approximating System near equilibrium $(0, 2)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = -2, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Equilibrium $(0, 2)$ is an attractive improper node

Example 4. Linear dynamics near $(1,1)$

The Linear Approximating System near equilibrium $(1,1)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

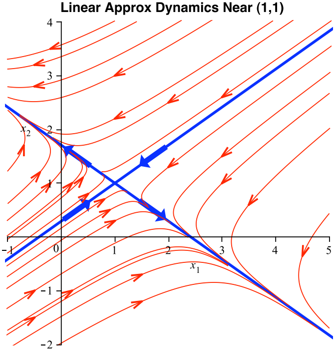
Eigenvalues & Eigenvectors:

$$\lambda_1 = -1 + \sqrt{2} > 0$$

$$\vec{\mathbf{w}}_1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

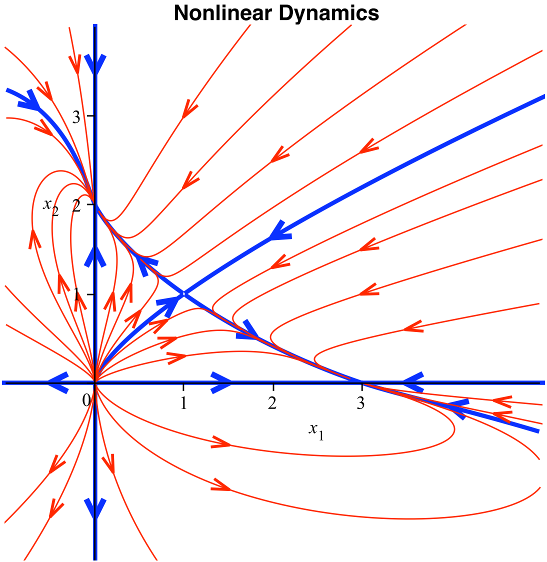
$$\lambda_2 = -1 - \sqrt{2} < 0$$

$$\vec{\mathbf{w}}_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

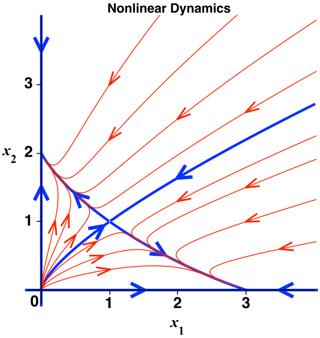


Equilibrium $(1,1)$ is a saddle

Example 4. Global phase portrait

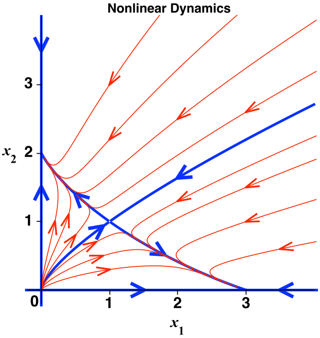


Example 4. Discussion



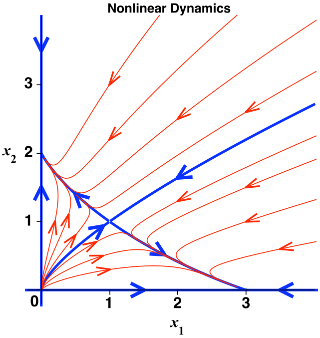
Example 4. Discussion

- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.



Example 4. Discussion

- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.
- The co-existence state $(1, 1)$ is unstable.

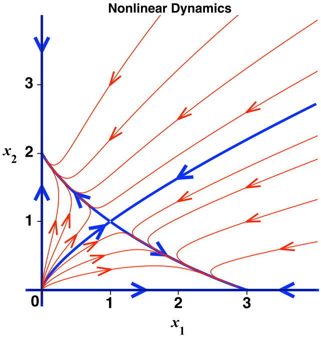


Example 4. Discussion

- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.

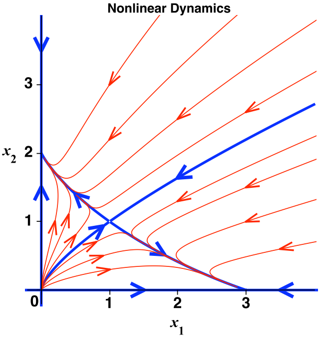
- The co-existence state $(1, 1)$ is unstable.

- Almost all positive solutions converge to either $(3, 0)$ or $(0, 2)$.



Example 4. Discussion

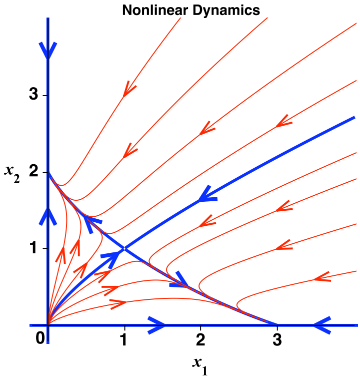
- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.
- The co-existence state $(1, 1)$ is unstable.
- Almost all positive solutions converge to either $(3, 0)$ or $(0, 2)$.
- A small difference in the initial conditions may make a huge difference in a species' destiny.



Example 4. (continued. Strong competition)

Question: Why is the co-existence unstable in this system?

Answer: Strong competition.



Example 4. (continued. Strong competition)

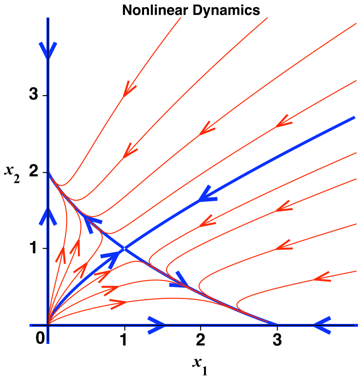
Question: Why is the co-existence unstable in this system?

Answer: Strong competition.

$$\begin{cases} x_1' = x_1(\ 3\ \ -x_1\ \ \boxed{-2x_2}\ ) \\ x_2' = x_2(\ 2\ \boxed{-x_1}\ \ -x_2\ \ ) \end{cases}$$

The competition terms

$\boxed{-2x_2}$ and $\boxed{-x_1}$



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The competition terms

$\boxed{-2x_2}$ and $\boxed{-x_1}$

ARE "STRONGER" THAN

the resource inhibition terms

$\boxed{-x_1}$ and $\boxed{-x_2}$