Solution Structures of 2nd Order Linear and Nonlinear Diff Eqs

[1] Given the fact that $y_1 = 0$ is a particular solution of a homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$:

(a) Can you produce two other solutions of this differential equation?

(b) Can you find all solutions of this differential equation?

[2] Suppose that $y_1 = -5e^{3\sin 2x}$ is a particular solution of a homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$.

(a) Can you produce two other solutions of this differential equation?

(b) Can you find all solutions of this differential equation?

[3] Suppose that $y_1 = -5e^{-2x}$ and $y_2 = e^{-2x}$ are particular solutions of a homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$.

(a) Can you produce two other solutions of this differential equation?

(b) Can you find all solutions of this differential equation?

[4] Suppose that $y_1 = -5e^{3x}$ and $y_2 = e^{-2x}$ are particular solutions of a homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$.

(a) Can you produce two other solutions of this differential equation?

(b) Can you find all solutions of this differential equation?

[5] Can some homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$ have the following two functions, $y_1 = e^{2x}$ and $y_2 = e^{-x}$, as particular solutions?

[6] Can some homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$ have the following three functions, $y_1 = e^{2x}$, $y_2 = e^{x}$, and $y_3 = xe^{x}$, as particular solutions?

[7] Can some homogeneous linear equation $y'' + b(x)y' + c(x)y = 0$ have the following three functions, $y_1 = -x^2 + \frac{1}{x}$, $y_2 = x^2 - \frac{2}{x}$, and $y_3 = 4x^2 - \frac{3}{x}$ as particular solutions?

[8] Suppose that $y_p = x^2$ satisfies $y'' + b(x)y' + c(x)y = f(x)$.

(a) Can you produce two other solutions of this differential equation?

(b) Can you find all solutions of this differential equation?

[9] Suppose that

$$\begin{cases} y_p = x^2 \text{ satisfies } y'' + b(x)y' + c(x)y = f(x), \\ \text{and } y_1 = e^{3x} \text{ satisfies } y'' + b(x)y' + c(x)y = 0. \end{cases}$$

(a) Can you produce two other solutions of the homogeneous equation $y'' + b(x)y' + c(x)y = 0$?

(b) Can you find all solutions of the homogeneous equation $y'' + b(x)y' + c(x)y = 0$?

(c) Can you produce two other solutions of the nonhomogeneous equation $y'' + b(x)y' + c(x)y = f(x)$?

(d) Can you find all solutions of the nonhomogeneous equation $y''+b(x)y'+c(x)y = f(x)$?

[10] Suppose that

$$\begin{cases} y_p = x^2 \text{ satisfies } y'' + b(x)y' + c(x)y = f(x), \\ \text{and } y_1 = e^{3x} \text{ and } y_2 = e^x \text{ satisfy } y'' + b(x)y' + c(x)y = 0. \end{cases}$$

(a) Can you produce two other solutions of the equation $y'' + b(x)y' + c(x)y = 0$?

(b) Can you find all solutions of the equation $y'' + b(x)y' + c(x)y = 0$?

(c) Can you produce two other solutions of the equation $y'' + b(x)y' + c(x)y = f(x)$?

(d) Can you find all solutions of the equation $y'' + b(x)y' + c(x)y = f(x)$?

[11] Suppose that $y_1 = x^2$, $y_2 = x + x^2$, and $y_3 = x^3$ are particular solutions of a nonhomogeneous linear equation $y'' + b(x)y' + c(x)y = f(x)$. Can you find all solutions of the nonhomogeneous equation $y'' + b(x)y' + c(x)y = f(x)$?

[12] Given the fact that $y_1 = e^x$ is a solution of a nonlinear differential equation $y'' = f(x, y)$, can you find all solutions of this equation?

[13] Given the fact that $y_1 = e^x$, $y_2 = -2e^{-x}$, and $y_3 = e^{3x}$ are solutions of a nonlinear differential equation $y'' = f(x, y)$, can you find all solutions of this equation?

[14] Given the fact that all four functions $y_1 = e^x$, $y_2 = -2e^{-x}$, $y_3 = 3e^x$, and $y_4 = -3e^{-x}$ are particular solutions of a nonlinear differential equation $y'' = f(x, y)$, can you find all solutions of this equation?

(See next page for answers)

Answers:

[1] (a) Not enough information. (b) Not enough information.

[2] (a) $Ce^{3\sin 2x}$ with any constant C is a solution. Answer: $0.3e^{3\sin 2x}, -7e^{3\sin 2x}, \cdots$
(b) Not enough information.

[3] (a) $0.3e^{-2x}, -7e^{-2x}, \cdots$
(b) Not enough information.

[4] (a) $0.3e^{3x} - 7e^{-2x}, 6e^{-2x}, \cdots$
(b) $y = C_1e^{3x} + C_2e^{-2x}$ where C_1 and C_2 are free parameters.

[5] Yes. Functions $y_1 = e^{3x}$ and $y_2 = e^{-x}$ both satisfy $y'' - 2y' - 3y = 0$.

Hint: Substituting $y_1 = e^{3x}$ and $y_2 = e^{-x}$ into $y'' + b(x)y' + c(x)y = 0$, we obtain $9 + 3b + c = 0, 1 - b + c = 0$. Now solve for b and c.

[6] Impossible.

Hint: Following the method used in the previous problem, plug $y_1 = e^{2x}$, $y_2 = e^x$ and $y_3 = xe^x$ into $y'' + b(x)y' + c(x)y = 0$. We obtain

$$\begin{cases} 4 + 2b + c = 0, \\ 1 + b + c = 0, \\ 2 + x + (1 + x)b + xc = 0. \end{cases}$$

This linear system for (b, c) has no solutions.

Remark: Another way to solve the problem is to notice that in general, a second order homogeneous linear diff eq $y'' + b(x)y' + c(x)y = 0$ can have at most two linearly independent solutions. The given functions $y_1 = e^{2x}$, $y_2 = e^x$ and $y_3 = xe^x$, however, are linearly independent. Thus, no second order homogeneous linear equation can have all these three functions as solutions.

[7] Yes, y_1, y_2 and y_3 all satisfy $y'' - \dfrac{2}{x^2}y = 0$.

Hint: The three functions y_1, y_2 and y_3 are linearly dependent. They all are linear combinations of x^2 and $1/x$.

[8] (a) Not enough information. (b) Not enough information.

[9] (a) $6e^{3x}, -0.45e^{3x}, \cdots$
(b) Not enough information.
(c) $x^2 + Ce^{3x}$ with any constant C is a solution. Answer: $x^2 + e^{3x}, x^2 - 0.45e^{3x}, \cdots$
(d) Not enough information.

[10] (a) $6e^{3x}, -5e^x, -2e^{3x} + 7e^x, \cdots$
(b) $y = C_1e^{3x} + C_2e^x$ where C_1 and C_2 are free parameters.
(c) $x^2 + e^{3x}, x^2 - 5e^x, x^2 - 2e^{3x} + 7e^x, \cdots$
(d) $y = x^2 + C_1e^{3x} + C_2e^x$ where C_1 and C_2 are free parameters.

[11] $y = x^2 + C_1 x + C_2(x^3 - x^2)$ where C_1 and C_2 are free parameters.

Hint: Since y_1, y_2, and y_3 all satisfy the same nonhomegeneous equation $y'' + b(x)y' + c(x)y = f(x)$, one can verify that $y_2 - y_1$ and $y_3 - y_1$ satisfy the corresponding homogeneous equation $y'' + b(x)y' + c(x)y = 0$.

[12] Not enough information.

[13] Not enough information.

[14] Not enough information.