A projective plane, built from $\mathbb{R}^2$

This is a description of the real projective plane we discussed in class. Note that here I'm using $(-\pi/2, \pi/2]$, instead of the $[0, \pi)$ that I said in class. Either will work, but it's easier to do the details with $(-\pi/2, \pi/2]$.

The points of this geometry are $\mathbb{R}^2 \cup (-\pi/2, \pi/2]$. That is, a point is **either** an ordered pair of real numbers (x, y) **or** it is a real number $x \in [0, \pi)$. For example, $(0, 0)$, $\frac{\pi}{2}$, $(-3.24, \pi^2)$, and 1 are all points in this geometry.

All the lines except one in this geometry are given by sets

$$\left\{t \cdot (x_0, y_0) + (x_1, y_1) \,\middle|\, t \in \mathbb{R}\right\} \cup \{\arctan(y_0/x_0)\},$$

where (x_0, y_0) is not $(0, 0)$, and (x_0, y_0) and (x_1, y_1) are points in $\mathbb{R}^2$, and we declare that

$$\arctan(\infty) = \arctan(-\infty) = \pi/2.$$

You can think about this as follows: the first term is the description of a "normal line" in $\mathbb{R}^2$, and the point $\arctan(y_0/x_0)$ keeps track of the slope of the line. Another way of writing this is to break the lines into two categories: vertical and non-vertical lines. The vertical lines are of the form

$$\left\{(x_0, y) \,\middle|\, y \in \mathbb{R}\right\} \cup \left\{\frac{\pi}{2}\right\},$$

for a fixed value of $x_0 \in \mathbb{R}$, and the non-vertical lines are of the form

$$\left\{(x, y) \,\middle|\, y = mx + b\right\} \cup \{\arctan m\},$$

for a fixed value $m \in \mathbb{R}$, where m is the slope of the line.

For example, we might have the line $\left\{(0, y) \,\middle|\, y \in \mathbb{R}\right\} \cup \{\pi/2\}$, which corresponds to the y–axis, a vertical line. We might have non-vertical line $\left\{(x, y) \,\middle|\, y = x\right\} \cup \{\pi/4\}$, which corresponds to the line $y = x$. These two lines intersect, as usual, in the point $(0, 0)$. If we consider the vertical line $\left\{(1, y) \,\middle|\, y \in \mathbb{R}\right\} \cup \{\pi/2\}$, which corresponds to the line $x = 1$, then the two vertical lines intersect at the point $\pi/2$.

In fact, any two lines that we normally think of as not being parallel still intersect where we expect them to (their slopes m_1 and m_2 are distinct, so $\arctan m_1 \neq \arctan m_2$). Any two lines that we normally think of as being parallel now intersect at the point $\arctan m$. For example, all vertical lines intersect at the point $\pi/2$. This is the purpose of keeping track of the slope: to turn $\mathbb{R}^2$ from an incidence geometry into a projective plane, **every twos lines need to intersect**. Since we started with parallel lines that didn't intersect, we "forced" them to intersect in the simplest way possible: add a new point that they have in common. However, we don't want to add new intersection points between lines that **already** intersected, so we come up with the idea of using the slope as our extra point.

I mentioned before that we covered all the lines except for one of them. The last line is given by the set

$$(-\frac{\pi}{2}, \frac{\pi}{2}].$$

Sometimes called the “line at infinity”, it’s the line that we are required to add in so that this satisfies the axioms of an incidence geometry. Recall that (I1) requires a line between every two points. Well, $\pi/2$ and $\pi/4$ are points in this geometry, so there needs to be a line connecting them. There is no “normal” line that comes from $\mathbb{R}^2$ that is **both** vertical **and** has slope 1, so we have to add a new line to fix this. The easiest way to do it is to add a new line that connects all the extra “slope points” that we’ve added.

Alternatively, instead of thinking of your slopes as angles, you can think of your slopes as the value of $m = \tan\theta$. Then, your points in your plane will be $\mathbb{R}^2 \cup \mathbb{R} \cup \{\infty\}$, since you will need to add all possible values of $m \in \mathbb{R}$ (all non-vertical lines) plus the point ∞ (for vertical lines). The lines in this version will be non-vertical:

$$\big\{(x, y) \,\big|\, y = mx + b\big\} \cup \{m\},$$

vertical:

$$\big\{(x_0, y) \,\big|\, y \in \mathbb{R}\big\} \cup \{\infty\},$$

and the “line at infinity”:

$$\{m \in \mathbb{R}\} \cup \{\infty\}.$$