Math 130
Homework 3 – due February 18, 2020
Jamie Conway

Work on all the questions, but please only submit questions **1ab, 2a, 5, and 6** to be graded.

1. Recall that a number is *algebraic* if it is the root of a polynomial in $\mathbb{Q}[x]$. It is *algebraic of degree* n if the lowest-degree polynomial of which it is a root has degree n.

   (a) Prove that if a is algebraic of degree n, then $\sqrt{a}$ is algebraic of degree at most $2n$. Must it be exactly $2n$?

   (b) Prove that if a is algebraic of degree n, then a^2 is algebraic of degree at most n. Must it be exactly n?

   (c) Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2})(\sqrt{3})$, and conclude that this field has degree 4 over $\mathbb{Q}$.

2. Show that:

   (a) if L_1 and L_2 are two line segments whose endpoints have coordinates in a field F, then the intersection of L_1 and L_2 has coordinates in F.

   (b) if $(x-a)^2+(y-b)^2 = r^2$ and $(x-c)^2+(y-d)^2 = s^2$ are two circles, with $a, b, c, d, r, s \in F$, then a point (x, y) where the circles intersect is the root of a degree-2 polynomial with coefficients in F.

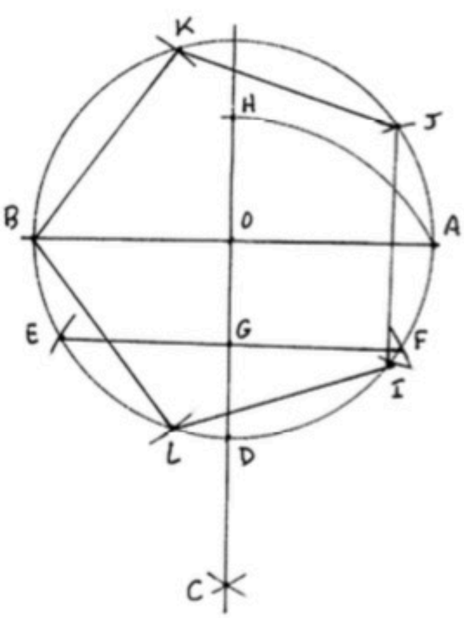
   Hint: replace one equation with the difference of the two equations to reduce the problem to the intersection of a circle and a straight line.

3. Prove that the construction below gives a regular pentagon.

1. Draw any line through *O*. Get *A*, *B*.
2. Circle *AB*.
3. Circle *BA*, get *C*.
4. *OC*, get *D*.
5. Circle *DO*. Get *E*, *F*.
6. *EF*, get *G*.
7. Circle *GA*, get *H*.
8. Circle center *A*, radius *OH*, get *I*, *J*.
9. Circle center *B*, radius *IJ*, get *K*, *L*.

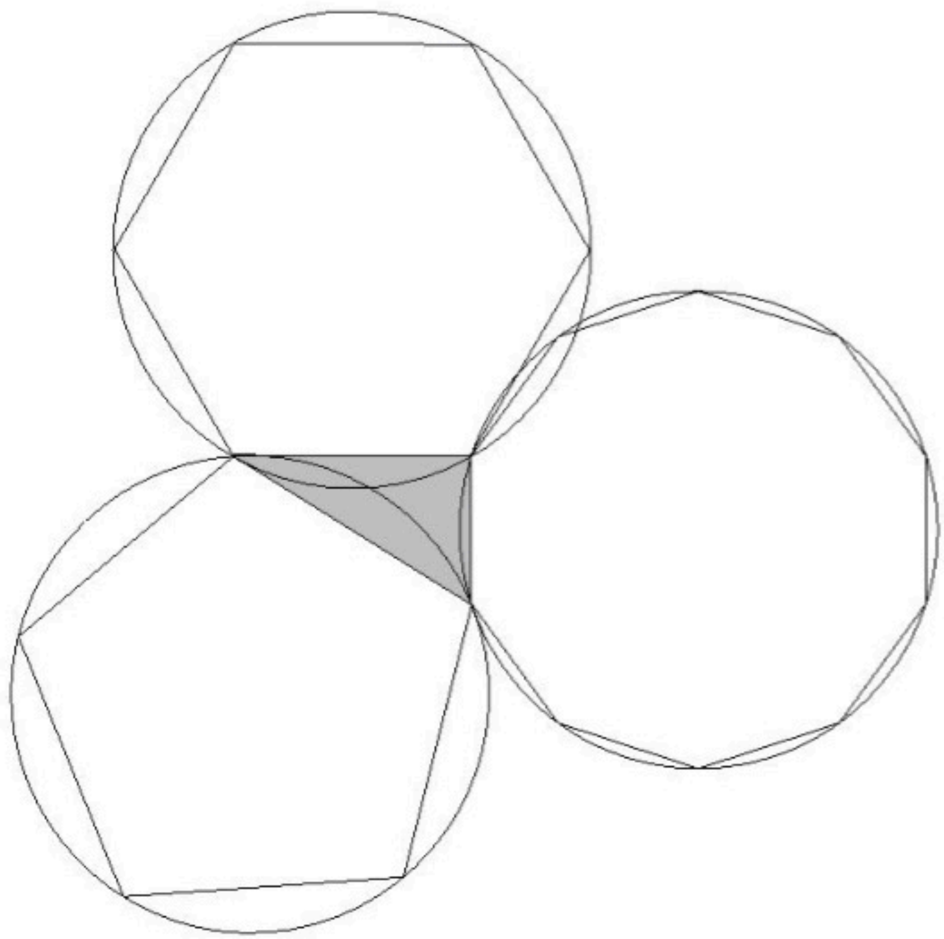
10–14. Draw *BK*, *KJ*, *JI*, *IL*, *LB*.

Then *BKJIL* is the required pentagon.



4. (a) Prove that the triangle formed by the sides of an inscribed hexagon, pentagon, and decagon is right-angled. See the illustration. You may use any high-school-level geometry in your proof.

   (b) Using (a), show that the segment AH in (3) has the same length as any side of the pentagon.



5. (a) Prove that the regular 9-gon is not constructible. You may use results that we proved in class (except the Gauss–Wantzel theorem).

   (b) More generally, prove that an angle of d degrees, where $0 < d < 180$ is an integer, is constructible if and only if d is a multiple of 3 (still without using the Gauss–Wantzel theorem).

6. Show, without using the Gauss–Wantzel theorem, that if you can construct a regular m-gon and a regular n-gon, where m and n are relatively prime, then you can construct a regular mn-gon.

7. Let $z \in \mathbb{C}$. Prove that the following are equivalent. Exposure to Galois theory will help!

   (a) z is constructible.

   (b) There is a sequence of degree 2 field extensions $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n$, where $z \in F_n$, and $F_i = F_{i-1}(z_i)$, $z_n = z$, and $z_i^2 \in F_{i-1}$.

   (c) $z \in F$, where F is a normal field extension of $\mathbb{Q}$. (Recall, F is a normal extension of $\mathbb{Q}$ if every $f \in F[x]$ has either no roots in F or splits into linear factors.)

   (d) The Galois group of the splitting field of z over $\mathbb{Q}$ is of order 2^k. (Recall that the splitting field of z is the smallest field containing all the roots of the minimal polynomial of z, and the Galois group of an extension F over $\mathbb{Q}$ is the group of automorphisms of F that fix $\mathbb{Q}$.)