Math 142

Homework 8 – Due April 3, 2018

Jamie Conway

1. Let K be the Klein bottle, T^2 the 2–dimensional torus, and $\mathbb{R}P^2$ the 2–dimensional projective plane.

   (a) Show that $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

   (b) Show that $T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

2. Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. We want to prove that there exists $x \in S^2$ such that $f(x) = f(-x)$.

   (a) Suppose no such x exists. Define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. Show that $g(-x) = -g(x)$.

   (b) Let $\alpha : [0,1] \to S^2$ be the path $\alpha(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ (that is, the equator), and consider the path $\sigma = g \circ \alpha$ in S^1. Show that $\sigma(t+\frac{1}{2}) = -\sigma(t)$ for all $t \in [0, \frac{1}{2}]$.

   (c) Assume $\sigma(0) = 1 \in S^1$ (this just simplifies the notation). Let the path $\widetilde{\sigma} : [0,1] \to \mathbb{R}$ be the unique lift of σ that has $\widetilde{\sigma}(0) = 0$. Show that $\widetilde{\sigma}(t + \frac{1}{2}) = \widetilde{\sigma}(t) + \frac{n}{2}$, for some odd integer $n \in \mathbb{Z}$ and $t \in [0, \frac{1}{2}]$.

   Hint: your proof will probably show that given a t, such an n exists. You also need to show that n is independent of t, by showing that it depends continuously on the value of t, and hence must be constant.

   (d) Show that $\widetilde{\sigma}(1) = n$, and hence that $[\sigma]$ represents a non-trivial element of $\pi_1(S^1)$.

   (e) Show that $[\alpha]$ is a trivial element of $\pi_1(S^2)$, and hence find a contradiction.

3. A *commutative division algebra structure* on $\mathbb{R}^n$ is a map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (which we write as multiplication $(a, b) \mapsto ab$, for $a, b \in \mathbb{R}^n$) that satisfies:

- $ab = ba$ for all $a, b \in \mathbb{R}^n$,
- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in \mathbb{R}^n$,
- $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in \mathbb{R}$ and $a, b \in \mathbb{R}^n$,
- Both $ax = b$ and $ya = b$ have a unique solution $x, y \in \mathbb{R}^n$ whenever $a \neq 0 \in \mathbb{R}^n$.

Our goal is to show that if $\mathbb{R}^n$ has a commutative division algebra structure, then $n = 1$ or $n = 2$ (note that $\mathbb{R}$ and $\mathbb{C}$ give examples in those dimensions).

(a) Let X be a closed n–manifold, Y a connected n–manifold, and let $f : X \to Y$ be an embedding (that is, $f : X \to f(X)$ is a homeomorphism). Show that f is surjective (that is, f is in fact a homeomorphism).

Hint: show that $f(X)$ is a closed subset of Y, and then show that $f(X)$ is an open subset of Y. The former uses topological properties of X and Y. The latter uses the fact that X is an n–manifold, and that n is the same as the dimension of Y, and the following fact that you can assume: given any embedding $f : B^n \to \mathbb{R}^n$, the set $\mathbb{R}^n \backslash f(\partial B^n)$ has two connected components, where the closure of one is compact and of the other is non-compact.

(b) Assume that $\mathbb{R}^n$ has a commutative division algebra structure, and $n > 2$. Show that $aa \neq 0$ whenever $a \neq 0 \in \mathbb{R}^n$.

(c) Define a map $f : S^{n-1} \to S^{n-1}$ by letting $f(a) = (aa)/||aa||$. This is a continuous map. Show that f induces a map $F : \mathbb{R}P^{n-1} \to S^{n-1}$ that is defined by $F(\{a, -a\}) = f(a)$.

(d) Show that F is injective (that is, show that $f(a) = f(b)$ if and only if $a = \pm b$).

(e) Show that F is a homeomorphism.

Hint: show that F is an embedding by noting that $\mathbb{R}P^{n-1}$ is compact and S^{n-1} is Hausdorff (and use a theorem from class/Armstrong). Then use (a).

(f) Show that $\mathbb{R}P^{n-1}$ is not homeomorphic to S^{n-1} for $n > 2$.

(g) (optional) It turns out that $\mathbb{C}$ is not the only commutative division algebra structure on $\mathbb{R}^2$. Define $(x, y) \cdot (z, w) = (xz - yw, -xw - yz)$ (that is, $a \cdot b = \overline{ab}$, for complex numbers a, b). Show that this gives a commutative division algebra structure on $\mathbb{R}^2$ that has no identity element.

(h) (optional) Show that $\mathbb{C}$ is the unique commutative division algebra structure on $\mathbb{R}^2$ with an identity element.