Math 142

Homework 7 – Due March 20, 2018

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1. Do the following problems from Armstrong:
   - Page 111 #33, #39 (just answer #39 regarding #33)
2. Prove the following statements that we gave in class:
   (a) If X is an n–manifold without boundary, and $U \subseteq X$ is open, then U (with the subspace topology) is an n–manifold.
   (b) (optional) If X is an n–manifold without boundary, and G is a group that acts *super nicely* on X, then X/G is an n–manifold. (A group G acts *super nicely* on X if it acts on X, has no fixed points, and additionally for all compact sets $A \subseteq X$, there are only finitely many g such that $f_g(A) \cap A \neq \varnothing$. This is needed to prove that X/G is Hausdorff.)
   (c) If X is an n–manifold with non-empty boundary, then ∂X is an $(n-1)$–manifold without boundary. You may assume the fact that the interior and boundary of X are disjoint sets.
3. (a) Let S be a surface (*ie.* a 2–manifold) without boundary. Show that $S\#S^2$ is homeomorphic to S. If it helps, you may assume the fact that for any embedding $f : D^2 \to S^2$, there is a homeomorphism $g : S^2 \to S^2$ such that the image of $g \circ f$ is the upper hemisphere of S^2.
   (b) What is $S^1\#S^1$? What is $S^1\#\mathbb{R}$? What is $\mathbb{R}\#\mathbb{R}$?
4. Consider the set $\mathfrak{M}_n$ of closed n–manifolds with finitely many connected components. We also define the empty set to be an n–manifold. Define a relation on M_n by setting $X \sim Y$ if and only if there exists a compact $(n+1)$–manifold W with boundary such that $\partial W \cong X \coprod Y$. In particular, we have that $X \sim \varnothing$ if and only if there is an $(n+1)$–manifold W with boundary $\partial W \cong X$.
   (a) Show that this relation on $\mathfrak{M}_n$ is an equivalence relation. *Hint: for reflexivity, consider* $W = X \times [0,1]$.
   (b) Let $\mathfrak{N}_n$ be the set of equivalence classes. Define an operation $+$ on $\mathfrak{N}_n$ by setting $[X] + [Y] = [X \coprod Y]$. Show that this is well-defined (that is, if $[X] = [X']$ and $[Y] = [Y']$, then $[X] + [Y] = [X'] + [Y']$.)
   (c) Show $[X] + [X] = [\varnothing]$ for all $[X]$.
   (d) Show that $(\mathfrak{N}_n, +)$ is an Abelian group with identity element $[\varnothing]$. This group is called the *(unoriented) cobordism group of dimension n*.