MATH 104
Midterm 2 – Study Guide
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The second class midterm will be on Thursday, March 23, in class. You will have the entire class time to complete the midterm. The midterm will cover all the definitions and results that we have covered in class (see the course website for a list of sections in Ross).

The midterm is closed book – no textbook, no notes, no calculators, etc. You will not be expected to know the proofs of the theorems by heart, but you should be comfortable with the definitions (such as continuity, uniform continuity, pointwise and uniform convergence, and sequences of functions), know the theorem statements, and be able to apply the theorems to problems (such as you did in the homework).

The midterm will consist of a few short questions/multiple choice, followed by three longer questions that will require you to construct a proof. Your proof should be similar in rigour to those given in the textbook or in homework solutions.

To prepare for the midterm, make sure that you understand all the homework questions and the results and examples done in class. Each section of Ross that we covered contains questions that were not assigned as homework but that are excellent practice for the midterm. Additionally, below are some problems similar to possible midterm questions.

1. Consider $f(x) = x^2(2-x)$ and $g(x) = |f(x)|$, defined for all $x \in \mathbb{R}$.
   (a) Use the ϵ-δ definition of continuity to prove that g is continuous at $x = 2$.
   (b) Prove that there are at least four solutions to the equation $g(x) = \frac{1}{2}$.
2. Let $f(x)$ be given by $f(x) = 1 - x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$. Show that f is continuous at $x = 1$.
3. Let f be a continuous function on $[a, b]$. Show that the function
$$g(x) = \sup\left\{ f(y) \,\middle|\, y \in [a, x] \right\},$$
   defined for $x \in [a, b]$, is a continuous function that is monotonically increasing.
4. Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $\mathbb{R}$.
5. Show that $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, 2]$ using the ϵ-δ definition. Do the same for $g(x) = \frac{5x}{2x-1}$ on $[1, \infty)$.
6. Let f be uniformly continuous on S, where S is a bounded set. Show that f is a bounded function. Use this to show that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Hint: use Bolzano–Weierstrass and the fact that uniformly continuous functions take Cauchy sequences to Cauchy sequences.

7. Consider a sequence (f_n) that converges uniformly to f on a set S, and let g be a bounded function on S. Let $h_n = g \cdot f_n$. Prove that (h_n) converges uniformly to $g \cdot f$ on S.

8. Let (f_n) be a sequence of bounded functions on S, and suppose that $f_n \to f$ uniformly on S. Prove that f is a bounded function on S.

9. Let (f_n) be a sequence of continuous functions defined on $[0, 1]$. Suppose that $f_n \to f$ uniformly on $[0, 1]$. Let

$$M = \sup \left\{ |f_n(x)| \;\middle|\; n \in \mathbb{N},\ x \in [0, 1] \right\}.$$

Prove that M is finite.

10. Consider the function g on $[0, \infty)$ defined by

$$g(x) = \begin{cases} x(1 - x) & \text{for } x \in [0, 1], \\ 0 & \text{for } x > 1. \end{cases}$$

Define a sequence of functions (f_n) on $[0, 1]$ by letting $f_n(x) = g(nx)$.

   (a) What is the maximum value of g, and at which x value is it attained?

   (b) Find a function f defined on $[0, 1]$ such that $f_n \to f$ pointwise. Prove your assertion.

   (c) Does $f_n \to f$ uniformly on $[0, 1]$? Prove your assertion.

11. Let f be a real-valued function on $(0, 1)$. Define a sequence of functions (f_n) on $(0, 1)$ by letting

$$f_n(x) = \begin{cases} \alpha & \text{for } x \in (0, \frac{1}{n}), \\ f(x) & \text{for } x \in [\frac{1}{n}, 1) \end{cases},$$

where $\alpha \in \mathbb{R}$ is a constant.

   (a) Prove that $f_n \to f$ pointwise.

   (b) Prove that $f_n \to f$ uniformly if and only if, for every $\epsilon > 0$, there is some $\delta > 0$, such that $|f(x) - \alpha| < \epsilon$ whenever $0 < x < \delta$.

12. Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\frac{y}{1 + y^2} \right)^n$$

converges for all $y \in \mathbb{R}$.

Hint: Show that $\left| \frac{y}{1+y^2} \right| < 1$, *and use the comparison test.*

13. Find the radius of convergence of each of the following series:

$$\sum \frac{x^n}{n\sqrt{n}}, \qquad \sum 4^n x^{2n+1}, \qquad \sum x^{n^2}.$$

14. True/False (and for your own benefit, explain):

   (a) If f is continuous on $(0, \infty)$, and f is uniformly continuous on $(0, a)$ for all $a > 0$, then f must be uniformly continuous on $(0, \infty)$.

   (b) Suppose (a_k) is a sequence of real numbers that satisfies $\lim |a_k|^{1/k} = 1$. Then, the infinite series $\sum a_k x^k$ must converge pointwise on $(-1, 1)$ but cannot converge uniformly on $(-1, 1)$.

   (c) Consider a sequence (f_n) of continuous functions on $[0, 1]$, and suppose that $f_n \to f$ pointwise on $[0, 1]$. If

   $$\lim \left(\int_0^1 f_n(x)\, dx \right) = \int_0^1 f(x)\, dx,$$

   then $f_n \to f$ uniformly on $[0, 1]$.

   (d) If a sequence (f_n) converges uniformly on $(0, 1)$ to f, then (f_n^3) converges uniformly on $(0, 1)$ to f^3.

   (e) Let $a \in (0, 1)$ be a fixed number. Suppose that the sequence of functions (g_n) on $[0, a]$ satisfies $|g_n(x)| \leq x^n$ for all $x \in [0, a]$, and for all $n \in \mathbb{N}$. Then $\sum g_n(x)$ is a uniformly convergent infinite series of functions.