MATH 104
Midterm 1 – Study Guide
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The first class midterm will be on Tuesday, February 21, in class. You will have the entire class time to complete the midterm. The midterm will cover all the definitions and results that we have covered in class (see the course website for a list of sections in Ross). Although we covered a few sections in Lebl, these won't be explicitly tested.

The midterm is closed book – no textbook, no notes, no calculators, etc. You will not be expected to know the proofs of the theorems by heart, but you should be comfortable with the definitions (such as suprema/infima, convergence/divergence, lim sup/inf, and infinite series), and be able to apply the theorems to problems (such as you did in the homework).

The midterm will consist of a few short questions/multiple choice, followed by three longer questions that will require you to construct a proof. Your proof should be similar in rigour to those given in the textbook or in homework solutions.

To prepare for the midterm, make sure that you understand all the homework questions and the results and examples done in class. Each section of Ross that we covered contains questions that were not assigned as homework but that are excellent practice for the midterm. Additionally, below are some problems similar to possible midterm questions.

1. Let (a_n) and (b_n) be sequences. Suppose $a_n \to a$ and $b_n \to b$. If $a_n \geq b_n$ for all n, prove that $a \geq b$. Note: Either a or b (or both) could be infinite, and your proof should deal with those cases separately, if necessary.

2. Prove that the infinite series $\sum \frac{2^k}{\sqrt{k!}}$ converges.

3. Let (a_n) and (b_n) be two bounded sequences.

   (a) Define a sequence (c_n) by $c_n = a_n + b_n$. Prove that $\limsup a_n + \limsup b_n \geq \limsup c_n$.

   (b) If $a_n \geq 0$ and $b_n \geq 0$, let $d_n = a_n b_n$. Show that $(\limsup a_n)(\limsup b_n) \geq \limsup d_n$.

4. Prove that $\lim \frac{\cos n}{\sqrt{n}} = 0$, using the definition of the limit (meaning, using ϵ and N, not any limit theorems).

5. Let (s_n) be a sequence diverging to ∞ and let (t_n) be a bounded sequence. Prove that $\lim (s_n + t_n) = \infty$.

6. Determine whether the infinite series $\sum_{k=2}^{\infty} \frac{1}{n^2 + (-1)^n}$ converges or diverges. You should include the details of your argument, including stating the test that you use (if any) and showing how its hypotheses are satisfied.

7. Define a sequence (x_n) by setting $x_1 = 1$, and defining $x_{n+1} = \sqrt{2x_n}$ for $n \geq 2$. Prove that (x_n) is an increasing sequence. Is it bounded? Does it converge? If so, find $\lim x_n$.

8. Let $S \subseteq \mathbb{R}$ be a non-empty subset, and suppose $\sup S = a \notin S$. Show that there exists a sequence (a_n) with $a_n \in S$ for every n such that $a_n \to a$.

9. Let $a \geq b > 0$. Show that $\lim (a^n + b^n)^{1/n} = a$. Hint: use the squeeze lemma.

10. Let (s_n) and (t_n) be two Cauchy sequences. Let a and b be two real constants. Show that the sequence (u_n) defined by $u_n = a \cdot s_n + b \cdot t_n$ is a Cauchy sequence only by using the definition (meaning, do not use the fact that (s_n) and (t_n) are convergent sequences, nor any limit theorems from Ross §9).