All very amusing, and we don't buy that explanation. A self-supporting pile of turtles is ludicrous, and not just because it's turtles. Each turtle being supported by a previous one just doesn't look like an explanation of how *the whole pile* stays up.

Very well. But now replace the Earth by the present state of the universe, and replace each turtle by the previous state of the universe. Oh, and change 'support' to 'cause'. Why does the universe exist? Because a previous one did. Why did that one exist? Because a previous one did. Did it all start a finite time in the past? No, it's *universes all the way back*.*

So a universe that has always existed is at least as puzzling as one that has not.

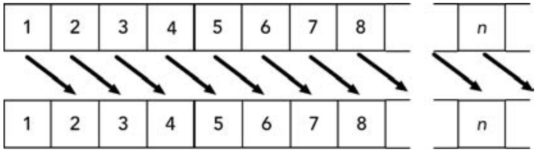
Hilbert's Hotel

Among the paradoxes concerning the infinite are a series of bizarre events at Hilbert's Hotel. David Hilbert was one of the world's leading mathematicians around 1900. He worked in the logical foundations of mathematics and took a particular interest in infinity. Anyway, Hilbert's Hotel has infinitely many rooms, numbered 1, 2, 3, 4, and so on – every positive integer.

One bank holiday weekend, the hotel was completely full. A traveller without a reservation arrived at reception wanting a room. In any finite hotel, no matter how big, the traveller would be out of luck – but not in Hilbert's Hotel.

'No problem, sir,' said the manager. 'I'll ask the person in Room 1 to move to Room 2, the person in Room 2 to move to Room 3, the person in Room 3 to move to Room 4, and so on. The person in Room n will move to Room $n+1$. Then Room 1 will be free, so I'll put you there.'

* Part of the appeal of the multiverse approach is that it revives the 'it's always been here' point of view. Our universe hasn't, but the surrounding multiverse has. It's multiverses all the way back ...

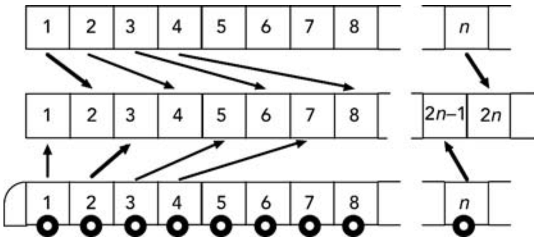


All move up one, and Room 1 is free.

This trick works in an infinite hotel. In a finite hotel it goes wrong, because the person in the room with the biggest number has nowhere to go. But in Hilbert's Hotel there is *no* biggest room number. Problem sorted.

Ten minutes later, an Infinity Tours coach arrived, with infinitely many passengers sitting in seats 1, 2, 3, 4, and so on.

'Well, I can't fit you in by asking every other guest to move up some number of places,' said the manager. 'Even if they all moved up a million places, that would only free up a million rooms.' He thought for a moment. 'Nevertheless, I can still fit you in. I'll ask the person in Room 1 to move to Room 2, the person in Room 2 to move to Room 4, the person in Room 3 to move to Room 6, and so on. The person in Room n will move to Room $2n$. That frees up all the odd-numbered rooms, so now I can put the person in Seat 1 of your bus into Room 1, the person in Seat 2 into Room 3, the person in Seat 3 into Room 5, and so on. The person in Seat n will move to Room $2n - 1$.'



How to accommodate an infinite bus-load.

However, the manager's troubles were still not over. Ten minutes later, he was horrified to see infinitely many Transfinity Travel buses arriving in his (infinite) car park.

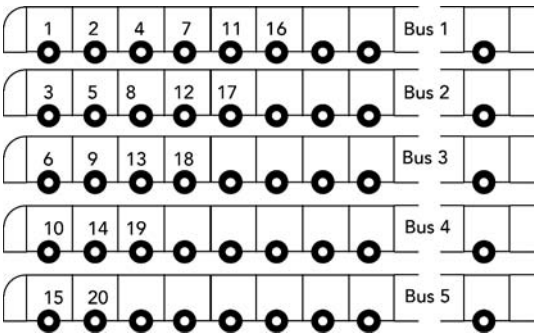
He rushed out to meet them. 'We're full – but I can *still* fit you all in!'

'How?' asked the driver of Bus 1.

'I'll reduce you to a problem I've already solved,' said the manager. 'I want you to move everyone into Bus 1.'

'But Bus 1 is full! And there are infinitely many other buses!'

'No problem. Line up all your buses side by side, and renumber all the seats using a diagonal order.'



The Manager's 'diagonal' order – the numbers 2–3, 4–5–6, 7–8–9–10, and so on slant to the left.

'What does that achieve?' asked the driver.

'Nothing – yet. But notice: each passenger, in each of your infinitely many buses, is assigned a new number. Every number occurs exactly once.'

'And your point is—?'

'Move each passenger to the seat in Bus 1 that corresponds to their new number.'

The driver did so. Then everyone was sitting in Bus 1, and all the other buses were empty – so they drove away.

'Now I've got a full hotel and just one extra bus-load,' said the manager. 'And I already know how to deal with that.'

Continuum Coaches

You won't be surprised to hear that the Hilbert's Hotel eventually ran into an accommodation problem that the manager could *not* solve. This time the hotel was completely empty – not that this ever seemed to make much difference. Then one of Cantor's Continuum Coaches stopped at the front door.

Georg Cantor was the first to sort out the mathematics of infinite sets. And he discovered something remarkable about the 'continuum' – the real number system. A *real number* is one that can be written as a decimal, which can either stop after finitely many digits, like 1.44, or go on for ever, like π. Here's what Cantor found.

The seats of the Continuum Coach were numbered using real numbers, not positive integers.

'Well,' the manager thought, 'one infinity is just like any other, right?' So he assigned passengers to rooms, and eventually the Hotel was full and the lobby was empty. The manager sighed with relief. 'Everyone has a room,' he said to himself.

Then a forlorn figure came in through the revolving doors.

'Good evening,' said the manager.

'My name is Mr Diagonal. Geddit? Missed-a-diagonal. You've missed me out, mate.'

'Well, I can always bump everyone up one room—'

'No, mate, you said "Everyone has a room" – I heard you. But I don't.'

'Nonsense! You've gone to your room, then nipped out the back and come in the front. I know your kind!'

'No, mate – I can *prove* I'm not in any of your rooms. Who's in Room 1?'

'I can't reveal personal information about guests.'

'What's the first decimal place of their coach seat?'

'I suppose I can reveal that. It's a 2.'

'My first digit is 3. So I'm not the person in Room 1, mate. Agreed?'

'Agreed.'

'What's the *second* decimal place of the coach seat of the person in Room 2?'

'It's a 7.'

'My second digit is 5. So I'm not the person in Room 2.'

'That makes sense.'

'Yeah, mate, and it goes on doing that. What's the *third* decimal place of the coach seat of the person in Room 3?'

'It's a 4.'

'My third digit is 8. So I'm not the person in Room 3.'

'Hmm. I think I see where this is headed.'

'Too right, mate. My *n*th digit is different from the *n*th digit of the person in Room *n*, *for every n*. So I'm not in Room *n*. Like I said, you left me out.'

'And like *I* said, I can always bump everyone up one place and fit you in.'

'No use, mate. There's infinitely many more just like me out there, sitting in your car park waiting for a room. However you assign passengers to rooms, there's going to be someone on the coach whose *n*th digit is different from the *n*th digit of the person in Room *n*, for every *n*. Hordes of them, in fact. You'll always miss people out.'

Now, you understand that Cantor didn't quite write his proof in those terms, but that was the basic idea. He proved that the infinite set of real numbers can't be matched, one for one, with the infinite set of whole numbers. Some infinities are bigger than others.

A Puzzling Dissection

'Why are you hacking that chessboard to bits?' asked Innumeratus.

'I want to show you something about areas,' said Mathophila. 'What's the area of the chessboard if each square has area one square unit?'

Innumeratus thought about this, and because he was better