MATH 104

Final – Study Guide

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The final will be on Thursday, May 11, 8:10 - 11, in Hearst Mining 310 (our classroom). You will have 2 hours and 50 minutes to complete the final. The final will cover all the definitions and results that we have covered in class. In the list of testable material below, if a section is listed without any qualifications, then the entire sections is testable.

- **Chapter 1**: §1 (induction – bottom of page 2 to the end of the chapter), §3, §4, §5
- **Chapter 2**: §7, §8 (although no particular result in the section), §9, §10, §11 (up to but not including Theorem 11.7), §14, §15
- **Chapter 3**: §17, §18 (up to but not including Theorem 18.4), §19 (except Theorem 19.6), §20 (only Definition 20.3a, but see Definition 20.1 to understand it)
- **Chapter 4**: §23, §24, §25
- **Chapter 5**: §28, §29 (up to but not including Theorem 29.9), §31 (up to but not including Theorem 31.5)
- **Chapter 6**: §32 , §33 (up to but not including Discussion 33.10, and excluding Definition 33.7 and Theorem 33.8), §34 (up to but not including Theorem 34.4)
- **Metric Spaces**: §13 (up to an including Theorem 13.4, as well as Definition 13.6 and 13.8), §21 (Definition 21.1 and the statement of Theorem 21.3)

The final is closed book – no textbook, no notes, no calculators, etc. You will not be expected to know the proofs of the theorems by heart, but you should be comfortable with the definitions, know the theorem statements, and be able to apply the theorems to problems (such as you did in the homework and on your midterms).

The final will consist of a some short questions/multiple choice, followed by longer questions that will require you to construct a proof. Your proof should be similar in rigour to those given in the textbook, in homework solutions, and on your midterms.

To prepare for the final, make sure that you understand all the homework and midterm questions, as well as the results and examples done in class. Each section of Ross that we covered contains questions that were not assigned as homework but that are excellent practice for the final. The midterm study guides are still available for your perusal, and below are some additional problems similar to possible final questions. The distribution of questions here is not representative of the distribution of material on the final.

1. True/False

(a) An unbounded sequence can have no Cauchy subsequence.

(b) If $f_n \to f$ uniformly on S, then $f_n' \to f'$ uniformly on S.

(c) If f is differentiable on $[a, b]$, then it is integrable on $[a, b]$.

(d) There exists a monotone sequence of real numbers that has no limit in $\mathbb{R}$, but it has a convergent subsequence.

(e) If $\limsup s_n = \liminf s_n \in \mathbb{R}$, then (s_n) converges.

(f) If $f : (0, \infty) \to \mathbb{R}$ is continuous on $\mathbb{R}$ and uniformly continuous on $(0, a)$ for all $a > 0$, then f is uniformly continuous on $(0, \infty)$.

(g) For any two sequences (t_n) and (s_n), we have $\limsup(s_n t_n) = (\limsup s_n)(\limsup t_n)$.

(h) If $f_n \to f$ pointwise on $[0, 1]$, and

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f,$$

then $f_n \to f$ uniformly on $[0, 1]$.

(i) Let $a \in (0, 1)$ be a fixed number. Suppose that a sequence of functions (g_n) defined on $[0, a]$ satisfies $|g_n(x)| \leq x^n$ for all $x \in [0, a]$ and for all $n \in \mathbb{N}$. Then $\sum g_n(x)$ is a uniformly convergent infinite series.

(j) There exists a subset of $\mathbb{R}$ that is neither open nor closed.

(k) There exists a subset of $\mathbb{R}$ that is both open and closed (that is, clopen).

(l) The function $H(x)$ that is 0 on $(0, 1]$ and 1 on $[-1, 0]$ is integrable on $[-1, 1]$.

(m) Suppose (S, d) is a complete metric space, and (s_n) is a Cauchy sequence in this metric space. If $f : S \to \mathbb{R}$ is a continuous function, then $(f(s_n))$ be a Cauchy sequence of real numbers.

2. Let (x_n) be a sequence, and let $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$.

   (a) If $x_n \to a$, show that $y_n \to a$.

   (b) Give an example of a divergent sequence (x_n) such that (y_n) (as defined above) converges.

3. Let $x_n = \cos\left(\frac{n\pi}{3}\right)$. Find a convergent subsequence of (x_n) and compute $\limsup x_n$.

4. Determine (and prove your determination!) whether the following series converge or diverge:

$$\sum_{k=1}^{\infty} \frac{6^k}{k^k}, \qquad \sum_{k=1}^{\infty} \frac{1}{k + 1/2}$$

5. Suppose that f_n converges uniformly to f on S, and g is a bounded function on S. Prove that gf_n converges uniformly to gf on S.

6. Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to f on $[a, b]$. Show that if (x_n) is a sequence in $[a, b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

7. Prove that $d(x, y) = \min\{|x - y|, 1\}$ is a metric on $\mathbb{R}$. Is the set $(-5, 5)$ open with respect to this metric?

8. Find the Taylor series at 0 for e^{2x}. Determine its radius of convergence (you may use either the ratio or root test). Prove that the Taylor series converges to e^{2x} for every $x \in \mathbb{R}$.

9. Let f and g be continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists some point $c \in [a, b]$ such that $f(x) = g(x)$.

10. Determine (and prove your determination!) whether the following functions are uniformly continuous on the given intervals:
    (a) $f(x) = \sqrt{x}$ on $(0, 1)$.
    (b) $f(x) = \frac{x^2}{1+x}$ on $(1, \infty)$.

11. Let $f(x) = x^2$ for rational x and 0 for irrational x. Calculate the upper and lower Darboux integrals for f on $[0, 1]$. Is f integrable on $[0, 1]$?

12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and differentiable, and f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$. *Hint: you should make use of the mean value theorem.*

13. Construct an example of two functions f and g that are integrable on $[0, 1]$, where $f(x) \neq g(x)$ for any $x \in [0, 1]$, and yet $\int_0^1 f = \int_0^1 g$.

14. Let $f(x) = x|x|$. Compute f' and f'' and state their domains. Use the definition of derivative where appropriate.

15. Define
$$g_n(x) = \frac{x^{n+1} + 2}{(n^2 + \cos n)(x^n + 1)}.$$
    (a) For $a > 0$, prove that $\sum g_n(x)$ converges to a continuous function on $[0, a]$.
    (b) Prove that $\sum g_n(x)$ converges to a continuous function on $[0, \infty)$.