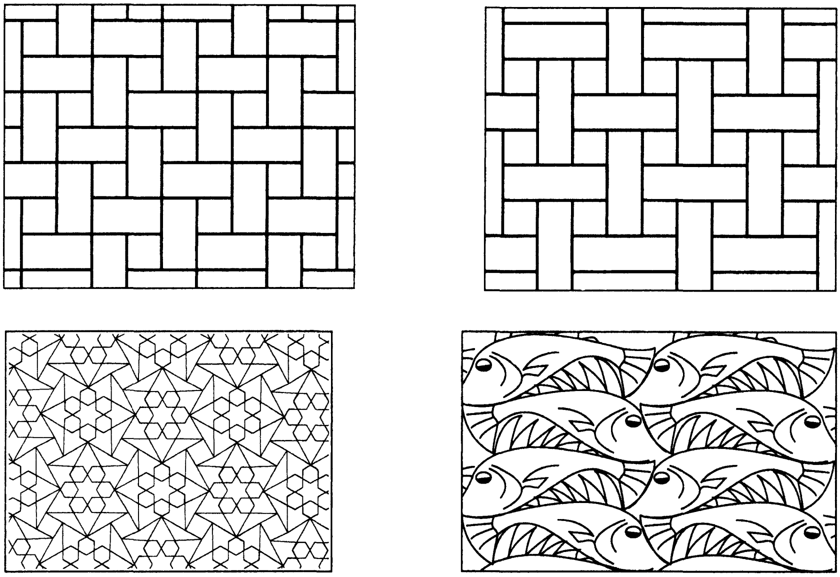

*6.4. Classify plane crystallographic groups with point group $D_1 = \{\bar{1}, \bar{r}\}$.

6.5. (a) Prove that if the point group of a two-dimensional crystallographic group G is C_6 or D_6, the translation group L is an equilateral triangular lattice.

(b) Classify those groups.

*6.6. Prove that symmetry groups of the figures in Figure 6.6.2 exhaust the possibilities.

Section 7 Abstract Symmetry: Group Operations

7.1. Let $G = D_4$ be the dihedral group of symmetries of the square.

(a) What is the stabilizer of a vertex? of an edge?

(b) G operates on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

7.2. The group M of isometries of the plane operates on the set of lines in the plane. Determine the stabilizer of a line.

7.3. The symmetric group S_3 operates on two sets U and V of order 3. Decompose the product set $U \times V$ into orbits for the "diagonal action" $g(u, v) = (gu, gv)$, when

(a) the operations on U and V are transitive,

(b) the operation on U is transitive, the orbits for the operation on V are $\{v_1\}$ and $\{v_2, v_3\}$.

7.4. In each of the figures in Exercise 6.3, find the points that have nontrivial stabilizers, and identify the stabilizers.

7.5. Let G be the group of symmetries of a cube, including the orientation-reversing symmetries. Describe the elements of G geometrically.

7.6. Let G be the group of symmetries of an equilateral triangular prism P, including the orientation-reversing symmetries. Determine the stabilizer of one of the rectangular faces of P and the order of the group.

7.7. Let $G = GL_n(\mathbb{R})$ operate on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this operation.

(b) What is the stabilizer of e_1?

7.8. Decompose the set $\mathbb{C}^{2\times 2}$ of 2×2 complex matrices into orbits for the following operations of $GL_2(\mathbb{C})$: **(a)** left multiplication, **(b)** conjugation.

7.9. **(a)** Let S be the set $\mathbb{R}^{m\times n}$ of real $m\times n$ matrices, and let $G = GL_m(\mathbb{R})\times GL_n(\mathbb{R})$. Prove that the rule $(P, Q) * A = PAQ^{-1}$ define an operation of G on S.

(b) Describe the decomposition of S into G-orbits.

(c) Assume that $m \leq n$. What is the stabilizer of the matrix $[I \mid 0]$?

7.10. **(a)** Describe the orbit and the stabilizer of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ under conjugation in the general linear group $GL_n(\mathbb{R})$.

(b) Interpreting the matrix in $GL_2(\mathbb{F}_5)$, find the order of the orbit.

7.11. Prove that the only subgroup of order 12 of the symmetric group S_4 is the alternating group A_4.

Section 8 The Operation on Cosets

8.1. Does the rule $P * A = PAP^t$ define an operation of GL_n on the set of $n\times n$ matrices?

8.2. What is the stabilizer of the coset $[aH]$ for the operation of G on G/H?

8.3. Exhibit the bijective map (6.8.4) explicitly, when G is the dihedral group D_4 and S is the set of vertices of a square.

8.4. Let H be the stabilizer of the index **1** for the operation of the symmetric group $G = S_n$ on the set of indices $\{\mathbf{1}, \ldots, \mathbf{n}\}$. Describe the left cosets of H in G and the map (6.8.4) in this case.

Section 9 The Counting Formula

9.1. Use the counting formula to determine the orders of the groups of rotational symmetries of a cube and of a tetrahedron.

9.2. Let G be the group of rotational symmetries of a cube, let G_v, G_e, G_f be the stabilizers of a vertex v, an edge e, and a face f of the cube, and let V, E, F be the sets of vertices, edges, and faces, respectively. Determine the formulas that represent the decomposition of each of the three sets into orbits for each of the subgroups.

9.3. Determine the order of the group of symmetries of a dodecahedron, when orientation-reversing symmetries such as reflections in planes are allowed.

9.4. Identify the group T' of all symmetries of a regular tetrahedron, including orientation-reversing symmetries.

9.5. Let F be a section of an I-beam, which one can think of as the product set of the letter I and the unit interval. Identify its group of symmetries, orientation-reversing symmetries included.

9.6. Identify the group of symmetries of a baseball, taking the seam (but not the stitching) into account and allowing orientation-reversing symmetries.

Section 10 Operations on Subsets

10.1. Determine the orders of the orbits for left multiplication on the set of subsets of order 3 of D_3.

10.2. Let S be a finite set on which a group G operates transitively, and let U be a subset of S. Prove that the subsets gU cover S evenly, that is, that every element of S is in the same number of sets gU.

10.3. Consider the operation of left multiplication by G on the set of its subsets. Let U be a subset such that the sets gU partition G. Let H be the unique subset in this orbit that contains 1. Prove that H is a subgroup of G.

Section 11 Permutation Representations

11.1. Describe all ways in which S_3 can operate on a set of four elements.

11.2. Describe all ways in which the tetrahedral group T can operate on a set of two elements.

11.3. Let S be a set on which a group G operates, and let H be the subset of elements g such that $gs = s$ for all s in S. Prove that H is a normal subgroup of G.

11.4. Let G be the dihedral group D_4 of symmetries of a square. Is the action of G on the vertices a faithful action? on the diagonals?

11.5. A group G operates faithfully on a set S of five elements, and there are two orbits, one of order 3 and one of order 2. What are the possible groups?
Hint: Map G to a product of symmetric groups.

11.6. Let $F = \mathbb{F}_3$. There are four one-dimensional subspaces of the space of column vectors F^2. List them. Left multiplication by an invertible matrix permutes these subspaces. Prove that this operation defines a homomorphism $\varphi: GL_2(F) \to S_4$. Determine the kernel and image of this homomorphism.

11.7. For each of the following groups, find the smallest integer n such that the group has a faithful operation on a set of order n: **(a)** D_4, **(b)** D_6, **(c)** the quaternion group H.

11.8. Find a bijective correspondence between the multiplicative group $\mathbb{F}_p^\times$ and the set of automorphisms of a cyclic group of order p.

11.9. Three sheets of rectangular paper S_1, S_2, S_3 are made into a stack. Let G be the group of all symmetries of this configuration, including symmetries of the individual sheets as well as permutations of the set of sheets. Determine the order of G, and the kernel of the map $G \to S_3$ defined by the permutations of the set $\{S_1, S_2, S_3\}$.

Section 12 Finite Subgroups of the Rotation Group

12.1. Explain why the groups of symmetries of the dodecahedron and the icosahedron are isomorphic.