Proof. The elements of $\overline{G}$ are the cosets of N, and they are also the fibres of the map φ (2.7.15). The map $\overline{\varphi}$ referred to in the theorem is the one that sends a nonempty fibre to its image: $\overline{\varphi}(\overline{x}) = \varphi(x)$. For any surjective map of sets $\varphi: G \to G'$, one can form the set $\overline{G}$ of fibres, and then one obtains a diagram as above, in which $\overline{\varphi}$ is the bijective map that sends a fibre to its image. When φ is a group homomorphism, $\overline{\varphi}$ is an isomorphism because $\overline{\varphi}(\overline{ab}) = \varphi(ab) = \varphi(a)\varphi(b) = \overline{\varphi}(\overline{a})\overline{\varphi}(\overline{b})$. □

Corollary 2.12.11 Let $\varphi: G \to G'$ be a group homomorphism with kernel N and image H'. The quotient group $\overline{G} = G/N$ is isomorphic to the image H'. □

Two quick examples: The image of the absolute value map $\mathbb{C}^\times \to \mathbb{R}^\times$ is the group of positive real numbers, and its kernel is the unit circle U. The theorem asserts that the quotient group $\mathbb{C}^\times/U$ is isomorphic to the multiplicative group of positive real numbers. The determinant is a surjective homomorphism $GL_n(\mathbb{R}) \to \mathbb{R}^\times$, whose kernel is the special linear group $SL_n(\mathbb{R})$. So the quotient $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$.

There are also theorems called the Second and the Third Isomorphism Theorems, though they are less important.

Es giebt also sehr viel verschiedene Arten von Größen,
welche sich nicht wohl herzehlen laßen;
und daher entstehen die derschiedene Theile der Mathematic,
deren eine jegliche mit einer besondern Art von Größen beschäftiget ist.

—Leonhard Euler

EXERCISES

Section 1 Laws of Composition

1.1. Let S be a set. Prove that the law of composition defined by $ab = a$ for all a and b in S is associative. For which sets does this law have an identity?

1.2. Prove the properties of inverses that are listed near the end of the section.

1.3. Let $\mathbb{N}$ denote the set $\{1, 2, 3, \ldots, \}$ of natural numbers, and let $s: \mathbb{N} \to \mathbb{N}$ be the *shift* map, defined by $s(n) = n + 1$. Prove that s has no right inverse, but that it has infinitely many left inverses.

Section 2 Groups and Subgroups

2.1. Make a multiplication table for the symmetric group S_3.

2.2. Let S be a set with an associative law of composition and with an identity element. Prove that the subset consisting of the invertible elements in S is a group.

2.3. Let x, y, z, and w be elements of a group G.

(a) Solve for y, given that $xyz^{-1}w = 1$.

(b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?

2.4. In which of the following cases is H a subgroup of G?

(a) $G = GL_n(\mathbb{C})$ and $H = GL_n(\mathbb{R})$.
(b) $G = \mathbb{R}^\times$ and $H = \{1, -1\}$.
(c) $G = \mathbb{Z}^+$ and H is the set of positive integers.
(d) $G = \mathbb{R}^\times$ and H is the set of positive reals.
(e) $G = GL_2(\mathbb{R})$ and H is the set of matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, with $a \neq 0$.

2.5. In the definition of a subgroup, the identity element in H is required to be the identity of G. One might require only that H have an identity element, not that it need be the same as the identity in G. Show that if H has an identity at all, then it is the identity in G. Show that the analogous statement is true for inverses.

2.6. Let G be a group. Define an *opposite group* G° with law of composition $a * b$ as follows: The underlying set is the same as G, but the law of composition is $a * b = ba$. Prove that G° is a group.

Section 3 Subgroups of the Additive Group of Integers

3.1. Let $a = 123$ and $b = 321$. Compute $d = \gcd(a, b)$, and express d as an integer combination $ra + bs$.

3.2. Prove that if a and b are positive integers whose sum is a prime p, their greatest common divisor is 1.

3.3. **(a)** Define the greatest common divisor of a set $\{a_1, \dots, a_n\}$ of n integers. Prove that it exists, and that it is an integer combination of $a_1, \dots, a_n$.
(b) Prove that if the greatest common divisor of $\{a_1, \dots, a_n\}$ is d, then the greatest common divisor of $\{a_1/d, \dots, a_n/d\}$ is 1.

Section 4 Cyclic Groups

4.1. Let a and b be elements of a group G. Assume that a has order 7 and that $a^3b = ba^3$. Prove that $ab = ba$.

4.2. An nth root of unity is a complex number z such that $z^n = 1$.

(a) Prove that the nth roots of unity form a cyclic subgroup of $\mathbb{C}^\times$ of order n.
(b) Determine the product of all the nth roots of unity.

4.3. Let a and b be elements of a group G. Prove that ab and ba have the same order.

4.4. Describe all groups G that contain no proper subgroup.

4.5. Prove that every subgroup of a cyclic group is cyclic. Do this by working with exponents, and use the description of the subgroups of $\mathbb{Z}^+$.

4.6. **(a)** Let G be a cyclic group of order 6. How many of its elements generate G? Answer the same question for cyclic groups of orders 5 and 8.
(b) Describe the number of elements that generate a cyclic group of arbitrary order n.

4.7. Let x and y be elements of a group G. Assume that each of the elements x, y, and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G, and that it has order 4.

4.8. **(a)** Prove that the elementary matrices of the first and third types (1.2.4) generate $GL_n(\mathbb{R})$.

(b) Prove that the elementary matrices of the first type generate $SL_n(\mathbb{R})$. Do the 2×2 case first.

4.9. How many elements of order 2 does the symmetric group S_4 contain?

4.10. Show by example that the product of elements of finite order in a group need not have finite order. What if the group is abelian?

4.11. **(a)** Adapt the method of row reduction to prove that the transpositions generate the symmetric group S_n.

(b) Prove that, for $n \geq 3$, the three-cycles generate the alternating group A_n.

Section 5 Homomorphisms

5.1. Let $\varphi: G \to G'$ be a surjective homomorphism. Prove that if G is cyclic, then G' is cyclic, and if G is abelian, then G' is abelian.

5.2. Prove that the intersection $K \cap H$ of subgroups of a group G is a subgroup of H, and that if K is a normal subgroup of G, then $K \cap H$ is a normal subgroup of H.

5.3. Let U denote the group of invertible upper triangular 2×2 matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \rightsquigarrow a^2$. Prove that φ is a homomorphism, and determine its kernel and image.

5.4. Let $f:\mathbb{R}^+ \to \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

5.5. Prove that the $n \times n$ matrices that have the block form $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, with A in $GL_r(\mathbb{R})$ and D in $GL_{n-r}(\mathbb{R})$, form a subgroup H of $GL_n(\mathbb{R})$, and that the map $H \to GL_r(\mathbb{R})$ that sends $M \rightsquigarrow A$ is a homomorphism. What is its kernel?

5.6. Determine the center of $GL_n(\mathbb{R})$.

Hint: You are asked to determine the invertible matrices A that commute with every invertible matrix B. Do not test with a general matrix B. Test with elementary matrices.

Section 6 Isomorphisms

6.1. Let G' be the group of real matrices of the form $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$. Is the map $\mathbb{R}^+ \to G'$ that sends x to this matrix an isomorphism?

6.2. Describe all homomorphisms $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$. Determine which are injective, which are surjective, and which are isomorphisms.

6.3. Show that the functions $f = 1/x$, $g = (x-1)/x$ generate a group of functions, the law of composition being composition of functions, that is isomorphic to the symmetric group S_3.

6.4. Prove that in a group, the products ab and ba are conjugate elements.

6.5. Decide whether or not the two matrices $A = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ are conjugate elements of the general linear group $GL_2(\mathbb{R})$.

6.6. Are the matrices $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}$ conjugate elements of the group $GL_2(\mathbb{R})$? Are they conjugate elements of $SL_2(\mathbb{R})$?

6.7. Let H be a subgroup of G, and let g be a fixed element of G. The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, with h in H. Prove that gHg^{-1} is a subgroup of G.

6.8. Prove that the map $A \rightsquigarrow (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.

6.9. Prove that a group G and its opposite group G° (Exercise 2.6) are isomorphic.

6.10. Find all automorphisms of
(a) a cyclic group of order 10, **(b)** the symmetric group S_3.

6.11. Let a be an element of a group G. Prove that if the set $\{1, a\}$ is a normal subgroup of G, then a is in the center of G.

Section 7 Equivalence Relations and Partitions

7.1. Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some g in G is an equivalence relation on G.

7.2. An equivalence relation on S is determined by the subset R of the set $S \times S$ consisting of those pairs (a, b) such that $a \sim b$. Write the axioms for an equivalence relation in terms of the subset R.

7.3. With the notation of Exercise 7.2, is the intersection $R \cap R'$ of two equivalence relations R and R' an equivalence relation? Is the union?

7.4. A relation R on the set of real numbers can be thought of as a subset of the (x, y)-plane. With the notation of Exercise 7.2, explain the geometric meaning of the reflexive and symmetric properties.

7.5. With the notation of Exercise 7.2, each of the following subsets R of the (x, y)-plane defines a relation on the set $\mathbb{R}$ of real numbers. Determine which of the axioms (2.7.3) are satisfied: **(a)** the set $\{(s, s) \mid s \in \mathbb{R}\}$, **(b)** the empty set, **(c)** the locus $\{xy + 1 = 0\}$, **(d)** the locus $\{x^2y - xy^2 - x + y = 0\}$.

7.6. How many different equivalence relations can be defined on a set of five elements?

Section 8 Cosets

8.1. Let H be the cyclic subgroup of the alternating group A_4 generated by the permutation $(1\,2\,3)$. Exhibit the left and the right cosets of H explicitly.

8.2. In the additive group $\mathbb{R}^m$ of vectors, let W be the set of solutions of a system of homogeneous linear equations $AX = 0$. Show that the set of solutions of an inhomogeneous system $AX = B$ is either empty, or else it is an (additive) coset of W.

8.3. Does every group whose order is a power of a prime p contain an element of order p?

8.4. Does a group of order 35 contain an element of order 5? of order 7?

8.5. A finite group contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?

8.6. Let $\varphi: G \to G'$ be a group homomorphism. Suppose that $|G| = 18$, $|G'| = 15$, and that φ is not the trivial homomorphism. What is the order of the kernel?

8.7. A group G of order 22 contains elements x and y, where $x \neq 1$ and y is not a power of x. Prove that the subgroup generated by these elements is the whole group G.

8.8. Let G be a group of order 25. Prove that G has at least one subgroup of order 5, and that if it contains only one subgroup of order 5, then it is a cyclic group.

8.9. Let G be a finite group. Under what circumstances is the map $\varphi: G \to G$ defined by $\varphi(x) = x^2$ an automorphism of G?

8.10. Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

8.11. Let G and H be the following subgroups of $GL_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \right\}, H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

with x and y real and $x > 0$. An element of G can be represented by a point in the right half plane. Make sketches showing the partitions of the half plane into left cosets and into right cosets of H.

8.12. Let S be a subset of a group G that contains the identity element 1, and such that the left cosets aS, with a in G, partition G. Prove that S is a subgroup of G.

8.13. Let S be a set with a law of composition. A partition $\Pi_1 \cup \Pi_2 \cup \cdots$ of S is *compatible* with the law of composition if for all i and j, the product set

$$\Pi_i \Pi_j = \{xy \mid x \in \Pi_i, y \in \Pi_j\}$$

is contained in a single subset Π_k of the partition.

(a) The set $\mathbb{Z}$ of integers can be partitioned into the three sets [Pos], [Neg], [{0}]. Discuss the extent to which the laws of composition $+$ and $\times$ are compatible with this partition.

(b) Describe all partitions of the integers that are compatible with the operation $+$.

Section 9 Modular Arithmetic

9.1. For which integers n does 2 have a multiplicative inverse in $\mathbb{Z}/\mathbb{Z}n$?

9.2. What are the possible values of a^2 modulo 4? modulo 8?

9.3. Prove that every integer a is congruent to the sum of its decimal digits modulo 9.

9.4. Solve the congruence $2x \equiv 5$ modulo 9 and modulo 6.

9.5. Determine the integers n for which the pair of congruences $2x - y \equiv 1$ and $4x + 3y \equiv 2$ modulo n has a solution.

9.6. Prove the *Chinese Remainder Theorem*: Let a, b, u, v be integers, and assume that the greatest common divisor of a and b is 1. Then there is an integer x such that $x \equiv u$ modulo a and $x \equiv v$ modulo b.

Hint: Do the case $u = 0$ and $v = 1$ first.

9.7. Determine the order of each of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ when the matrix entries are interpreted modulo 3.

Section 10 The Correspondence Theorem

10.1. Describe how to tell from the cycle decomposition whether a permutation is odd or even.

10.2. Let H and K be subgroups of a group G.

(a) Prove that the intersection $xH \cap yK$ of two cosets of H and K is either empty or else is a coset of the subgroup $H \cap K$.

(b) Prove that if H and K have finite index in G then $H \cap K$ also has finite index in G.

10.3. Let G and G' be cyclic groups of orders 12 and 6, generated by elements x and y, respectively, and let $\varphi : G \to G'$ be the map defined by $\varphi(x^i) = y^i$. Exhibit the correspondence referred to in the Correspondence Theorem explicitly.

10.4. With the notation of the Correspondence Theorem, let H and H' be corresponding subgroups. Prove that $[G:H] = [G':H']$.

10.5. With reference to the homomorphism $S_4 \to S_3$ described in Example 2.5.13, determine the six subgroups of S_4 that contain K.

Section 11 Product Groups

11.1. Let x be an element of order r of a group G, and let y be an element of G' of order s. What is the order of (x, y) in the product group $G \times G'$?

11.2. What does Proposition 2.11.4 tell us when, with the usual notation for the symmetric group S_3, K and H are the subgroups $\langle y \rangle$ and $\langle x \rangle$?

11.3. Prove that the product of two infinite cyclic groups is not infinite cyclic.

11.4. In each of the following cases, determine whether or not G is isomorphic to the product group $H \times K$.

(a) $G = \mathbb{R}^\times$, $H = \{\pm 1\}$, $K = \{\text{positive real numbers}\}$.

(b) $G = \{\text{invertible upper triangular } 2\times 2 \text{ matrices}\}$, $H = \{\text{invertible diagonal matrices}\}$, $K = \{\text{upper triangular matrices with diagonal entries } 1\}$.

(c) $G = \mathbb{C}^\times$, $H = \{\text{unit circle}\}$, $K = \{\text{positive real numbers}\}$.

11.5. Let G_1 and G_2 be groups, and let Z_i be the center of G_i. Prove that the center of the product group $G_1 \times G_2$ is $Z_1 \times Z_2$.

11.6. Let G be a group that contains normal subgroups of orders 3 and 5, respectively. Prove that G contains an element of order 15.

11.7. Let H be a subgroup of a group G, let $\varphi: G \to H$ be a homomorphism whose restriction to H is the identity map, and let N be its kernel. What can one say about the product map $H \times N \to G$?

11.8. Let G, G', and H be groups. Establish a bijective correspondence between homomorphisms $\Phi: H \to G \times G'$ from H to the product group and pairs (φ, φ') consisting of a homomorphism $\varphi: H \to G$ and a homomorphism $\varphi': H \to G'$.

11.9. Let H and K be subgroups of a group G. Prove that the product set HK is a subgroup of G if and only if $HK = KH$.

Section 12 Quotient Groups

12.1. Show that if a subgroup H of a group G is not normal, there are left cosets aH and bH whose product is not a coset.

12.2. In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $*$ represents an arbitrary real number. Show that H is a subgroup of GL_3, that K is a normal subgroup of H, and identify the quotient group H/K. Determine the center of H.

12.3. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P that contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.

12.4. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly. Is G/H isomorphic to G?

12.5. Let G be the group of upper triangular real matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, with a and d different from zero. For each of the following subsets, determine whether or not S is a subgroup, and whether or not S is a normal subgroup. If S is a normal subgroup, identify the quotient group G/S.

(i) S is the subset defined by $b = 0$.
(ii) S is the subset defined by $d = 1$.
(iii) S is the subset defined by $a = d$.

Miscellaneous Problems

M.1. Describe the column vectors $(a, c)^t$ that occur as the first column of an integer matrix A whose inverse is also an integer matrix.

M.2. **(a)** Prove that every group of even order contains an element of order 2.
(b) Prove that every group of order 21 contains an element of order 3.

M.3. Classify groups of order 6 by analyzing the following three cases:

(i) G contains an element of order 6.
(ii) G contains an element of order 3 but none of order 6.
(iii) All elements of G have order 1 or 2.

M.4. A *semigroup* S is a set with an associative law of composition and with an identity. Elements are not required to have inverses, and the Cancellation Law need not hold. A semigroup S is said to be generated by an element s if the set $\{1, s, s^2, \ldots\}$ of nonnegative powers of s is equal to S. Classify semigroups that are generated by one element.

M.5. Let S be a finite semigroup (see Exercise M.4) in which the Cancellation Law 2.2.3 holds. Prove that S is a group.

***M.6.** Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ be points in k-dimensional space $\mathbb{R}^k$. A *path* from a to b is a continuous function on the unit interval $[0, 1]$ with values in $\mathbb{R}^k$, a function $X: [0, 1] \to \mathbb{R}^k$, sending $t \rightsquigarrow X(t) = (x_1(t), \ldots, x_k(t))$, such that $X(0) = a$ and $X(1) = b$. If S is a subset of $\mathbb{R}^k$ and if a and b are in S, define $a \sim b$ if a and b can be joined by a path lying entirely in S.

(a) Show that $\sim$ is an equivalence relation on S. Be careful to check that any paths you construct stay within the set S.

(b) A subset S is *path connected* if $a \sim b$ for any two points a and b in S. Show that every subset S is partitioned into path-connected subsets with the property that two points in different subsets cannot be connected by a path in S.

(c) Which of the following loci in $\mathbb{R}^2$ are path-connected: $\{x^2 + y^2 = 1\}$, $\{xy = 0\}$, $\{xy = 1\}$?

***M.7.** The set of $n \times n$ matrices can be identified with the space $\mathbb{R}^{n \times n}$. Let G be a subgroup of $GL_n(\mathbb{R})$. With the notation of Exercise M.6, prove:

(a) If A, B, C, D are in G, and if there are paths in G from A to B and from C to D, then there is a path in G from AC to BD.

(b) The set of matrices that can be joined to the identity I forms a normal subgroup of G. (It is called the *connected component* of G.)

***M.8.** **(a)** The group $SL_n(\mathbb{R})$ is generated by elementary matrices of the first type (see Exercise 4.8). Use this fact to prove that $SL_n(\mathbb{R})$ is path-connected.

(b) Show that $GL_n(\mathbb{R})$ is a union of two path-connected subsets, and describe them.

M.9. (*double cosets*) Let H and K be subgroups of a group G, and let g be an element of G. The set $HgK = \{x \in G \mid x = hgk \text{ for some } h \in H, k \in K\}$ is called a *double coset*. Do the double cosets partition G?

M.10. Let H be a subgroup of a group G. Show that the double cosets (see Exercise M.9)

$$HgH = \{h_1 g h_2 | h_1, h_2 \in H\}$$

are the left cosets gH if and only if H is normal.

***M.11.** Most invertible matrices can be written as a product $A = LU$ of a lower triangular matrix L and an upper triangular matrix U, where in addition all diagonal entries of U are 1.

(a) Explain how to compute L and U when the matrix A is given.

(b) Prove uniqueness, that there is at most one way to write A as such a product.

(c) Show that every invertible matrix can be written as a product LPU, where L, U are as above and P is a permutation matrix.

(d) Describe the double cosets LgU (see Exercise M.9).

M.12. (*postage stamp problem*) Let a and b be positive, relatively prime integers.

(a) Prove that every sufficiently large positive integer n can be obtained as $ra + sb$, where r and s are positive integers.

(b) Determine the largest integer that is not of this form.

M.13. (*a game*) The starting position is the point $(1, 1)$, and a permissible "move" replaces a point (a, b) by one of the points $(a + b, b)$ or $(a, a + b)$. So the position after the first move will be either $(2, 1)$ or $(1, 2)$. Determine the points that can be reached.

M.14. (*generating $SL_2(\mathbb{Z})$*) Prove that the two matrices

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

generate the group $SL_2(\mathbb{Z})$ of all *integer* matrices with determinant 1. Remember that the subgroup they generate consists of all elements that can be expressed as products using the four elements E, E', E^{-1}, E'^{-1}.

Hint: Do not try to write a matrix directly as a product of the generators. Use row reduction.

M.15. (*the semigroup generated by elementary matrices*) Determine the semigroup S (see Exercise M.4) of matrices A that can be written as a product, of arbitrary length, each of whose terms is one of the two matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Show that every element of S can be expressed as such a product in exactly one way.

M.16. [1](*the homophonic group: a mathematical diversion*) By definition, English words have the same pronunciation if their phonetic spellings in the dictionary are the same. The homophonic group $\mathcal{H}$ is generated by the letters of the alphabet, subject to the following relations: English words with the same pronunciation represent equal elements of the group. Thus *be* = *bee*, and since $\mathcal{H}$ is a group, we can cancel *be* to conclude that $e = 1$. Try to determine the group $\mathcal{H}$.

[1] I learned this problem from a paper by Mestre, Schoof, Washington and Zagier.

EXERCISES

Section 1 Symmetry of Plane Figures

1.1. Determine all symmetries of Figures 6.1.4, 6.1.6, and 6.1.7.

Section 3 Isometries of the Plane

3.1. Verify the rules (6.3.3).

3.2. Let m be an orientation-reversing isometry. Prove algebraically that m^2 is a translation.

3.3. Prove that a linear operator on $\mathbb{R}^2$ is a reflection if and only if its eigenvalues are 1 and -1, and the eigenvectors with these eigenvalues are orthogonal.

3.4. Prove that a conjugate of a glide reflection in M is a glide reflection, and that the glide vectors have the same length.

3.5. Write formulas for the isometries (6.3.1) in terms of a complex variable $z = x + iy$.

3.6. **(a)** Let s be the rotation of the plane with angle $\pi/2$ about the point $(1, 1)^t$. Write the formula for s as a product $t_a\rho_\theta$.

(b) Let s denote reflection of the plane about the vertical axis $x = 1$. Find an isometry g such that $grg^{-1} = s$, and write s in the form $t_a\rho_\theta r$.

Section 4 Finite Groups of Orthogonal Operators on the Plane

4.1. Write the product $x^2yx^{-1}y^{-1}x^3y^3$ in the form x^iy^j in the dihedral group D_n.

4.2. **(a)** List all subgroups of the dihedral group D_4, and decide which ones are normal.

(b) List the proper normal subgroups N of the dihedral group D_{15}, and identify the quotient groups D_{15}/N.

(c) List the subgroups of D_6 that do not contain x^3.

4.3. **(a)** Compute the left cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10}.

(b) Prove that H is normal and that D_{10}/H is isomorphic to D_5.

(c) Is D_{10} isomorphic to $D_5 \times H$?

Section 5 Discrete Groups of Isometries

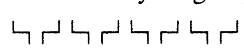
5.1. Let ℓ_1 and ℓ_2 be lines through the origin in $\mathbb{R}^2$ that intersect in an angle π/n, and let r_i be the reflection about ℓ_i. Prove that r_1 and r_2 generate a dihedral group D_n.

5.2. What is the crystallographic restriction for a discrete group of isometries whose translation group L has the form $\mathbb{Z}a$ with $a \neq 0$?

5.3. How many sublattices of index 3 are contained in a lattice L in $\mathbb{R}^2$?

5.4. Let (a, b) be a lattice basis of a lattice L in $\mathbb{R}^2$. Prove that every other lattice basis has the form $(a', b') = (a, b)P$, where P is a 2×2 integer matrix with determinant ± 1.

5.5. Prove that the group of symmetries of the frieze pattern ◁◁◁◁◁◁◁ is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

5.6. Let G be the group of symmetries of the frieze pattern  . Determine the point group $\overline{G}$ of G, and the index in G of its subgroup of translations.