constant term a_0 of f. This small loop won't wind around the origin at all. But as we just saw, $f(C_r)$ winds n times around the origin if r is large enough. The only explanation for this is that for some intermediate radius r', $f(C_{r'})$ passes through the origin. This means that for some point α on the circle $C_{r'}$, $f(\alpha) = 0$. Then α is a root of f.

I don't consider this algebra,
but this doesn't mean that algebraists can't do it.

—Garrett Birkhoff

EXERCISES

Section 1 Examples of Fields

1.1. Let R be an integral domain that contains a field F as subring and that is finite-dimensional when viewed as vector space over F. Prove that R is a field.

1.2. Let F be a field, not of characteristic 2, and let $x^2 + bx + c = 0$ be a quadratic equation with coefficients in F. Prove that if δ is an element of F such that $\delta^2 = b^2 - 4c$, $x = (-b + \delta)/2a$ solves the quadratic equation in F. Prove also that if the discriminant $b^2 - 4c$ is not a square, the polynomial has no root in F.

1.3. Which subfields of $\mathbb{C}$ are dense subsets of $\mathbb{C}$?

Section 2 Algebraic and Transcendental Elements

2.1. Let α be a complex root of the polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a + b\alpha + c\alpha^2$, with a, b, c in $\mathbb{Q}$.

2.2. Let $f(x) = x^n - a_{n-1}x^{n-1} + \cdots \pm a_0$ be an irreducible polynomial over F, and let α be a root of f in an extension field K. Determine the element α^{-1} explicitly in terms of α and of the coefficients a_i.

2.3. Let $\beta = \omega\sqrt[3]{2}$, where $\omega = e^{2\pi i/3}$, and let $K = \mathbb{Q}(\beta)$. Prove that the equation $x_1^2 + \cdots + x_k^2 = -1$ has no solution with x_i in K.

Section 3 The Degree of a Field Extension

3.1. Let F be a field, and let α be an element that generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

3.2. Prove that the polynomial $x^4 + 3x + 3$ is irreducible over the field $\mathbb{Q}[\sqrt[3]{2}]$.

3.3. Let $\zeta_n = e^{2\pi i/n}$. Prove that $\zeta_5 \notin \mathbb{Q}(\zeta_7)$.

3.4. Let $\zeta_n = e^{2\pi i/n}$. Determine the irreducible polynomial over $\mathbb{Q}$ and over $\mathbb{Q}(\zeta_3)$ of
(a) ζ_4, **(b)** ζ_6, **(c)** ζ_8, **(d)** ζ_9, **(e)** ζ_{10}, **(f)** ζ_{12}.

3.5. Determine the values of n such that ζ_n has degree at most 3 over $\mathbb{Q}$.

3.6. Let a be a positive rational number that is not a square in $\mathbb{Q}$. Prove that $\sqrt[4]{a}$ has degree 4 over $\mathbb{Q}$.

3.7. **(a)** Is i in the field $\mathbb{Q}(\sqrt[4]{-2})$? **(b)** Is $\sqrt[3]{5}$ in the field $\mathbb{Q}(\sqrt[3]{2})$?

3.8. Let α and β be complex numbers. Prove that if $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers, then α and β are also algebraic numbers.

3.9. Let α and β be complex roots of irreducible polynomials $f(x)$ and $g(x)$ in $\mathbb{Q}[x]$. Let $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$. Prove that $f(x)$ is irreducible in $L[x]$ if and only if $g(x)$ is irreducible in $K[x]$.

3.10. A field extension K/F is an *algebraic extension* if every element of K is algebraic over F. Let K/F and L/K be algebraic field extensions. Prove that L/F is an algebraic extension.

Section 4 Finding the Irreducible Polynomial

4.1. Let $K = \mathbb{Q}(\alpha)$, where α is a root of $x^3 - x - 1$. Determine the irreducible polynomial for $\gamma = 1 + \alpha^2$ over $\mathbb{Q}$.

4.2. Determine the irreducible polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the following fields. **(a)** $\mathbb{Q}$, **(b)** $\mathbb{Q}(\sqrt{5})$, **(c)** $\mathbb{Q}(\sqrt{10})$, **(d)** $\mathbb{Q}(\sqrt{15})$.

4.3. With reference to Example 15.4.4**(b)**, determine the irreducible polynomial for $\gamma = \alpha_1 + \alpha_2$ over $\mathbb{Q}$.

Section 5 Constructions with Ruler and Compass

5.1. Express $\cos 15°$ in terms of real square roots.

5.2. Prove that the regular pentagon can be constructed by ruler and compass
(a) by field theory, **(b)** by finding an explicit construction.

5.3. Decide whether or not the regular 9-gon is constructible by ruler and compass.

5.4. Is it possible to construct a square whose area is equal to that of a given triangle?

5.5. Referring to the proof of Proposition 15.5.5, suppose that the discriminant D is negative. Determine the line that appears at the end of the proof geometrically.

5.6. Thinking of the plane as the complex plane, describe the set of constructible points as complex numbers.

Section 6 Adjoining Roots

6.1. Let F be a field of characteristic zero, let f' denote the derivative of a polynomial f in $F[x]$, and let g be an irreducible polynomial that is a common divisor of f and f'. Prove that g^2 divides f.

6.2. **(a)** Let F be a field of characteristic zero. Determine all square roots of elements of F that a quadratic extension of the form $F(\sqrt{a})$ contains.

(b) Classify quadratic extensions of $\mathbb{Q}$.