Section 5 Adjoining Elements

5.1. Let $f = x^4 + x^3 + x^2 + x + 1$ and let α denote the residue of x in the ring $R = \mathbb{Z}[x]/(f)$. Express $(\alpha^3 + \alpha^2 + \alpha)(\alpha^5 + 1)$ in terms of the basis $(1, \alpha, \alpha^2, \alpha^3)$ of R.

5.2. Let a be an element of a ring R. If we adjoin an element α with the relation $\alpha = a$, we expect to get a ring isomorphic to R. Prove that this is true.

5.3. Describe the ring obtained from $\mathbb{Z}/12\mathbb{Z}$ by adjoining an inverse of 2.

5.4. Determine the structure of the ring R' obtained from $\mathbb{Z}$ by adjoining an element α satisfying each set of relations.

(a) $2\alpha = 6, 6\alpha = 15$, **(b)** $2\alpha - 6 = 0, \alpha - 10 = 0$, **(c)** $\alpha^3 + \alpha^2 + 1 = 0, \alpha^2 + \alpha = 0$.

5.5. Are there fields F such that the rings $F[x]/(x^2)$ and $F[x]/(x^2 - 1)$ are isomorphic?

5.6. Let a be an element of a ring R, and let R' be the ring $R[x]/(ax-1)$ obtained by adjoining an inverse of a to R. Let α denote the residue of x (the inverse of a in R').

(a) Show that every element β of R' can be written in the form $\beta = \alpha^k b$, with b in R.
(b) Prove that the kernel of the map $R \to R'$ is the set of elements b of R such that $a^n b = 0$ for some $n > 0$.
(c) Prove that R' is the zero ring if and only if a is nilpotent (see Exercise 3.9).

5.7. Let F be a field and let $R = F[t]$ be the polynomial ring. Let R' be the ring extension $R[x]/(tx - 1)$ obtained by adjoining an inverse of t to R. Prove that this ring can be identified as the ring of *Laurent polynomials*, which are finite linear combinations of powers of t, negative exponents included.

Section 6 Product Rings

6.1. Let $\varphi: \mathbb{R}[x] \to \mathbb{C} \times \mathbb{C}$ be the homomorphism defined by $\varphi(x) = (1, i)$ and $\varphi(r) = (r, r)$ for r in $\mathbb{R}$. Determine the kernel and the image of φ.

6.2. Is $\mathbb{Z}/(6)$ isomorphic to the product ring $\mathbb{Z}/(2) \times \mathbb{Z}/(3)$? Is $\mathbb{Z}/(8)$ isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$?

6.3. Classify rings of order 10.

6.4. In each case, describe the ring obtained from the field $\mathbb{F}_2$ by adjoining an element α satisfying the given relation:

(a) $\alpha^2 + \alpha + 1 = 0$, **(b)** $\alpha^2 + 1 = 0$, **(c)** $\alpha^2 + \alpha = 0$.

6.5. Suppose we adjoin an element α satisfying the relation $\alpha^2 = 1$ to the real numbers $\mathbb{R}$. Prove that the resulting ring is isomorphic to the product $\mathbb{R} \times \mathbb{R}$.

6.6. Describe the ring obtained from the product ring $\mathbb{R} \times \mathbb{R}$ by inverting the element $(2, 0)$.

6.7. Prove that in the ring $\mathbb{Z}[x]$, the intersection $(2) \cap (x)$ of the principal ideals (2) and (x) is the principal ideal $(2x)$, and that the quotient ring $R = \mathbb{Z}[x]/(2x)$ is isomorphic to the subring of the product ring $\mathbb{F}_2[x] \times \mathbb{Z}$ of pairs $(f(x), n)$ such that $f(0) \equiv n$ modulo 2.

6.8. Let I and J be ideals of a ring R such that $I + J = R$.

(a) Prove that $IJ = I \cap J$ (see Exercise 3.13).
(b) Prove the *Chinese Remainder Theorem*: For any pair a, b of elements of R, there is an element x such that $x \equiv a$ modulo I and $x \equiv b$ modulo J. (The notation $x \equiv a$ modulo I means $x - a \in I$.)

(c) Prove that if $IJ = 0$, then R is isomorphic to the product ring $(R/I)\times(R/J)$.
(d) Describe the idempotents corresponding to the product decomposition in **(c)**.

Section 7 Fractions

7.1. Prove that a domain of finite order is a field.

7.2. Let R be a domain. Prove that the polynomial ring $R[x]$ is a domain, and identify the units in $R[x]$.

7.3. Is there a domain that contains exactly 15 elements?

7.4. Prove that the field of fractions of the formal power series ring $F[[x]]$ over a field F can be obtained by inverting the element x. Find a neat description of the elements of that field (see Exercise 11.2.1).

7.5. A subset S of a domain R that is closed under multiplication and that does not contain 0 is called a *multiplicative set*. Given a multiplicative set S, define S-fractions to be elements of the form a/b, where b is in S. Show that the equivalence classes of S-fractions form a ring.

Section 8 Maximal Ideals

8.1. Which principal ideals in $\mathbb{Z}[x]$ are maximal ideals?

8.2. Determine the maximal ideals of each of the following rings:
(a) $\mathbb{R}\times\mathbb{R}$, **(b)** $\mathbb{R}[x]/(x^2)$, **(c)** $\mathbb{R}[x]/(x^2-3x+2)$, **(d)** $\mathbb{R}[x]/(x^2+x+1)$.

8.3. Prove that the ring $\mathbb{F}_2[x]/(x^3+x+1)$ is a field, but that $\mathbb{F}_3[x]/(x^3+x+1)$ is not a field.

8.4. Establish a bijective correspondence between maximal ideals of $\mathbb{R}[x]$ and points in the upper half plane.

Section 9 Algebraic Geometry

9.1. Let I be the principal ideal of $\mathbb{C}[x, y]$ generated by the polynomial y^2+x^3-17. Which of the following sets generate maximal ideals in the quotient ring $R = \mathbb{C}[x, y]/I$? $(x-1, y-4)$, $(x+1, y+4)$, (x^3-17, y^2).

9.2. Let $f_1, \ldots, f_r$ be complex polynomials in the variables $x_1, \ldots, x_n$, let V be the variety of their common zeros, and let I be the ideal of the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ that they generate. Define a homomorphism from the quotient ring $\overline{R} = R/I$ to the ring $\mathcal{R}$ of continuous, complex-valued functions on V.

9.3. Let $U = \{f_i(x_1, \ldots, x_m) = 0\}$, $V = \{g_j(y_1, \ldots, y_n) = 0\}$ be varieties in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Show that the variety defined by the equations $\{f_i(x) = 0, g_j(y) = 0\}$ in x, y-space $\mathbb{C}^{m+n}$ is the product set $U\times V$.

9.4. Let U and V be varieties in $\mathbb{C}^n$. Prove that the union $U\cup V$ and the intersection $U\cap V$ are varieties. What does the statement $U\cap V = \emptyset$ mean algebraically? What about the statement $U\cup V = \mathbb{C}^n$?

9.5. Prove that the variety of zeros of a set $\{f_1, \ldots, f_r\}$ of polynomials depends only on the ideal that they generate.

9.6. Prove that every variety in $\mathbb{C}^2$ is the union of finitely many points and algebraic curves.

9.7. Determine the points of intersection in $\mathbb{C}^2$ of the two loci in each of the following cases:
(a) $y^2-x^3+x^2=1,\ x+y=1$, **(b)** $x^2+xy+y^2=1,\ x^2+2y^2=1$,
(c) $y^2=x^3,\ xy=1$, **(d)** $x+y^2=0,\ y+x^2+2xy^2+y^4=0$.