EXERCISES

Section 1 Definition of a Ring

1.1. Prove that $7 + \sqrt[3]{2}$ and $\sqrt{3} + \sqrt{-5}$ are algebraic numbers.

1.2. Prove that, for $n \neq 0$, $\cos(2\pi/n)$ is an algebraic number.

1.3. Let $\mathbb{Q}[\alpha, \beta]$ denote the smallest subring of $\mathbb{C}$ containing the rational numbers $\mathbb{Q}$ and the elements $\alpha = \sqrt{2}$ and $\beta = \sqrt{3}$. Let $\gamma = \alpha + \beta$. Is $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$? Is $\mathbb{Z}[\alpha, \beta] = \mathbb{Z}[\gamma]$?

1.4. Let $\alpha = \frac{1}{2}i$. Prove that the elements of $\mathbb{Z}[\alpha]$ are dense in the complex plane.

1.5. Determine all subrings of $\mathbb{R}$ that are discrete sets.

1.6. Decide whether or not S is a subring of R, when

(a) S is the set of all rational numbers a/b, where b is not divisible by 3, and $R = \mathbb{Q}$,

(b) S is the set of functions which are linear combinations with integer coefficients of the functions $\{1, \cos nt, \sin nt\}$, $n \in \mathbb{Z}$, and R is the set of all real valued functions of t.

1.7. Decide whether the given structure forms a ring. If it is not a ring, determine which of the ring axioms hold and which fail:

(a) U is an arbitrary set, and R is the set of subsets of U. Addition and multiplication of elements of R are defined by the rules $A + B = (A \cup B) - (A \cap B)$ and $A \cdot B = A \cap B$.

(b) R is the set of continuous functions $\mathbb{R} \to \mathbb{R}$. Addition and multiplication are defined by the rules $[f + g](x) = f(x) + g(x)$ and $[f \circ g](x) = f(g(x))$.

1.8. Determine the units in: **(a)** $\mathbb{Z}/12\mathbb{Z}$, **(b)** $\mathbb{Z}/8\mathbb{Z}$, **(c)** $\mathbb{Z}/n\mathbb{Z}$.

1.9. Let R be a set with two laws of composition satisfying all ring axioms except the commutative law for addition. Use the distributive law to prove that the commutative law for addition holds, so that R is a ring.

Section 2 Polynomial Rings

2.1. For which positive integers n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 7x + 5$ in $[\mathbb{Z}/(n)][x]$?

2.2. Let F be a field. The set of all formal power series $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, with a_i in F, forms a ring that is often denoted by $F[[t]]$. By *formal* power series we mean that the coefficients form an arbitrary sequence of elements of F. There is no requirement of convergence. Prove that $F[[t]]$ is a ring, and determine the units in this ring.

Section 3 Homomorphisms and Ideals

3.1. Prove that an ideal of a ring R is a subgroup of the additive group R^+.

3.2. Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

3.3. Find generators for the kernels of the following maps:

(a) $\mathbb{R}[x, y] \to \mathbb{R}$ defined by $f(x, y) \rightsquigarrow f(0, 0)$,

(b) $\mathbb{R}[x] \to \mathbb{C}$ defined by $f(x) \rightsquigarrow f(2 + i)$,

(c) $\mathbb{Z}[x] \to \mathbb{R}$ defined by $f(x) \rightsquigarrow f(1 + \sqrt{2})$,

(d) $\mathbb{Z}[x] \to \mathbb{C}$ defined by $x \rightsquigarrow \sqrt{2} + \sqrt{3}$.
(e) $\mathbb{C}[x, y, z] \to \mathbb{C}[t]$ defined by $x \rightsquigarrow t$, $y \rightsquigarrow t^2$, $z \rightsquigarrow t^3$.

3.4. Let $\varphi: \mathbb{C}[x, y] \to \mathbb{C}[t]$ be the homomorphism that sends $x \rightsquigarrow t+1$ and $y \rightsquigarrow t^3 - 1$. Determine the kernel K of φ, and prove that every ideal I of $\mathbb{C}[x, y]$ that contains K can be generated by two elements.

3.5. The derivative of a polynomial f with coefficients in a field F is defined by the calculus formula $(a_n x^n + \cdots + a_1 x + a_0)' = n a_n x^{n-1} + \cdots + 1 a_1$. The integer coefficients are interpreted in F using the unique homomorphism $\mathbb{Z} \to F$.

(a) Prove the product rule $(fg)' = f'g + fg'$ and the chain rule $(f \circ g)' = (f' \circ g)g'$.
(b) Let α be an element of F. Prove that α is a multiple root of a polynomial f if and only if it is a common root of f and of its derivative f'.

3.6. An *automorphism* of a ring R is an isomorphism from R to itself. Let R be a ring, and let $f(y)$ be a polynomial in one variable with coefficients in R. Prove that the map $R[x, y] \to R[x, y]$ defined by $x \rightsquigarrow x + f(y)$, $y \rightsquigarrow y$ is an automorphism of $R[x, y]$.

3.7. Determine the automorphisms of the polynomial ring $\mathbb{Z}[x]$ (see Exercise 3.6).

3.8. Let R be a ring of prime characteristic p. Prove that the map $R \to R$ defined by $x \rightsquigarrow x^p$ is a ring homomorphism. (It is called the *Frobenius map*.)

3.9. **(a)** An element x of a ring R is called *nilpotent* if some power is zero. Prove that if x is nilpotent, then $1 + x$ is a unit.

(b) Suppose that R has prime characteristic $p \neq 0$. Prove that if a is nilpotent then $1 + a$ is *unipotent*, that is, some power of $1 + a$ is equal to 1.

3.10. Determine all ideals of the ring $F[[t]]$ of formal power series with coefficients in a field F (see Exercise 2.2).

3.11. Let R be a ring, and let I be an ideal of the polynomial ring $R[x]$. Let n be the lowest degree among nonzero elements of I. Prove or disprove: I contains a monic polynomial of degree n if and only if it is a principal ideal.

3.12. Let I and J be ideals of a ring R. Prove that the set $I + J$ of elements of the form $x + y$, with x in I and y in J, is an ideal. This ideal is called the *sum* of the ideals I and J.

3.13. Let I and J be ideals of a ring R. Prove that the intersection $I \cap J$ is an ideal. Show by example that the set of products $\{xy \mid x \in I, y \in J\}$ need not be an ideal, but that the set of finite sums $\sum x_\nu y_\nu$ of products of elements of I and J is an ideal. This ideal is called the *product ideal*, and is denoted by IJ. Is there a relation between IJ and $I \cap J$?

Section 4 Quotient Rings

4.1. Consider the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ that sends $x \rightsquigarrow 1$. Explain what the Correspondence Theorem, when applied to this map, says about ideals of $\mathbb{Z}[x]$.

4.2. What does the Correspondence Theorem tell us about ideals of $\mathbb{Z}[x]$ that contain $x^2 + 1$?

4.3. Identify the following rings: **(a)** $\mathbb{Z}[x]/(x^2 - 3, 2x + 4)$, **(b)** $\mathbb{Z}[i]/(2 + i)$, **(c)** $\mathbb{Z}[x]/(6, 2x - 1)$, **(d)** $\mathbb{Z}[x]/(2x^2 - 4, 4x - 5)$, **(e)** $\mathbb{Z}[x]/(x^2 + 3, 5)$.

4.4. Are the rings $\mathbb{Z}[x]/(x^2 + 7)$ and $\mathbb{Z}[x]/(2x^2 + 7)$ isomorphic?