9.5. Let F be a section of an I-beam, which one can think of as the product set of the letter I and the unit interval. Identify its group of symmetries, orientation-reversing symmetries included.

9.6. Identify the group of symmetries of a baseball, taking the seam (but not the stitching) into account and allowing orientation-reversing symmetries.

Section 10 Operations on Subsets

10.1. Determine the orders of the orbits for left multiplication on the set of subsets of order 3 of D_3.

10.2. Let S be a finite set on which a group G operates transitively, and let U be a subset of S. Prove that the subsets gU cover S evenly, that is, that every element of S is in the same number of sets gU.

10.3. Consider the operation of left multiplication by G on the set of its subsets. Let U be a subset such that the sets gU partition G. Let H be the unique subset in this orbit that contains 1. Prove that H is a subgroup of G.

Section 11 Permutation Representations

11.1. Describe all ways in which S_3 can operate on a set of four elements.

11.2. Describe all ways in which the tetrahedral group T can operate on a set of two elements.

11.3. Let S be a set on which a group G operates, and let H be the subset of elements g such that $gs = s$ for all s in S. Prove that H is a normal subgroup of G.

11.4. Let G be the dihedral group D_4 of symmetries of a square. Is the action of G on the vertices a faithful action? on the diagonals?

11.5. A group G operates faithfully on a set S of five elements, and there are two orbits, one of order 3 and one of order 2. What are the possible groups?
Hint: Map G to a product of symmetric groups.

11.6. Let $F = \mathbb{F}_3$. There are four one-dimensional subspaces of the space of column vectors F^2. List them. Left multiplication by an invertible matrix permutes these subspaces. Prove that this operation defines a homomorphism $\varphi: GL_2(F) \to S_4$. Determine the kernel and image of this homomorphism.

11.7. For each of the following groups, find the smallest integer n such that the group has a faithful operation on a set of order n: **(a)** D_4, **(b)** D_6, **(c)** the quaternion group H.

11.8. Find a bijective correspondence between the multiplicative group $\mathbb{F}_p^\times$ and the set of automorphisms of a cyclic group of order p.

11.9. Three sheets of rectangular paper S_1, S_2, S_3 are made into a stack. Let G be the group of all symmetries of this configuration, including symmetries of the individual sheets as well as permutations of the set of sheets. Determine the order of G, and the kernel of the map $G \to S_3$ defined by the permutations of the set $\{S_1, S_2, S_3\}$.

Section 12 Finite Subgroups of the Rotation Group

12.1. Explain why the groups of symmetries of the dodecahedron and the icosahedron are isomorphic.

EXERCISES

Section 1 Cayley's Theorem

1.1. Does the rule $g * x = xg^{-1}$ define an operation of G on G?

1.2. Let H be a subgroup of a group G. Describe the orbits for the operation of H on G by left multiplication.

Section 2 The Class Equation

2.1. Determine the centralizer and the order of the conjugacy class of

(a) the matrix $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ in $GL_2(\mathbb{F}_3)$, **(b)** the matrix $\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ in $GL_2(\mathbb{F}_5)$.

2.2. A group of order 21 contains a conjugacy class $C(x)$ of order 3. What is the order of x in the group?

2.3. A group G of order 12 contains a conjugacy class of order 4. Prove that the center of G is trivial.

2.4. Let G be a group, and let φ be the nth power map: $\varphi(x) = x^n$. What can be said about how φ acts on conjugacy classes?

2.5. Let G be the group of matrices of the form $\begin{bmatrix} x & y \\ & 1 \end{bmatrix}$, where $x, y \in \mathbb{R}$ and $x > 0$. Determine the conjugacy classes in G, and sketch them in the (x, y)-plane.

2.6. Determine the conjugacy classes in the group M of isometries of the plane.

2.7. Rule out as many as you can, as class equations for a group of order 10:

$1+1+1+2+5, \quad 1+2+2+5, \quad 1+2+3+4, \quad 1+1+2+2+2+2.$

2.8. Determine the possible class equations of nonabelian groups of order **(a)** 8, **(b)** 21.

2.9. Determine the class equation for the following groups: **(a)** the quaternion group, **(b)** D_4, **(c)** D_5, **(d)** the subgroup of $GL_2(\mathbb{F}_3)$ of invertible upper triangular matrices.

2.10. **(a)** Let A be an element of SO_3 that represents a rotation with angle π. Describe the centralizer of A geometrically.

(b) Determine the centralizer of the reflection r about the e_1-axis in the group M of isometries of the plane.

2.11. Determine the centralizer in $GL_3(\mathbb{R})$ of each matrix:

$$\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}.$$

***2.12.** Determine all finite groups that contain at most three conjugacy classes.

2.13. Let N be a normal subgroup of a group G. Suppose that $|N| = 5$ and that $|G|$ is an odd integer. Prove that N is contained in the center of G.

2.14. The class equation of a group G is $1 + 4 + 5 + 5 + 5$.

(a) Does G have a subgroup of order 5? If so, is it a normal subgroup?

(b) Does G have a subgroup of order 4? If so, is it a normal subgroup?

2.15. Verify the class equation (7.2.10) of $SL_2(\mathbb{F}_3)$.

2.16. Let $\varphi: G \to G'$ be a surjective group homomorphism, let C denote the conjugacy class of an element x of G, and let C' denote the conjugacy class in G' of its image $\varphi(x)$. Prove that φ maps C surjectively to C', and that $|C'|$ divides $|C|$.

2.17. Use the class equation to show that a group of order pq, with p and q prime, contains an element of order p.

2.18. Which pairs of matrices $\begin{bmatrix} 0 & 1 \\ -1 & d \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & d \end{bmatrix}$ are conjugate elements of **(a)** $GL_n(\mathbb{R})$, **(b)** $SL_n(\mathbb{R})$?

Section 3 p-Groups

3.1. Prove the Fixed Point Theorem (7.3.2).

3.2. Let Z be the center of a group G. Prove that if G/Z is a cyclic group, then G is abelian, and therefore $G = Z$.

3.3. A nonabelian group G has order p^3, where p is prime.

(a) What are the possible orders of the center Z?

(b) Let x be an element of G that isn't in Z. What is the order of its centralizer $Z(x)$?

(c) What are the possible class equations for G?

3.4. Classify groups of order 8.

Section 4 The Class Equation of the Icosahedral Group

4.1. The icosahedral group operates on the set of five inscribed cubes in the dodecahedron. Determine the stabilizer of one of the cubes.

4.2. Is A_5 the only proper normal subgroup of S_5?

4.3. What is the centralizer of an element of order 2 of the icosahedral group I?

4.4. **(a)** Determine the class equation of the tetrahedral group T.

(b) Prove that T has a normal subgroup of order 4, and no subgroup of order 6.

4.5. **(a)** Determine the class equation of the octahedral group O.

(b) This group contains two proper normal subgroups. Find them, show that they are normal, and show that there are no others.

4.6. **(a)** Prove that the tetrahedral group T is isomorphic to the alternating group A_4, and that the octahedral group O is isomorphic to the symmetric group S_4.

Hint: Find sets of four elements on which the groups operate.

(b) Two tetrahedra can be inscribed into a cube C, each one using half the vertices. Relate this to the inclusion $A_4 \subset S_4$.

4.7. Let G be a group of order n that operates nontrivially on a set of order r. Prove that if $n > r!$, then G has a proper normal subgroup.

4.8. **(a)** Suppose that the centralizer $Z(x)$ of a group element x has order 4. What can be said about the center of the group?

(b) Suppose that the conjugacy class $C(y)$ of an element y has order 4. What can be said about the center of the group?