EXERCISES

Section 1 Symmetry of Plane Figures

1.1. Determine all symmetries of Figures 6.1.4, 6.1.6, and 6.1.7.

Section 3 Isometries of the Plane

3.1. Verify the rules (6.3.3).

3.2. Let m be an orientation-reversing isometry. Prove algebraically that m^2 is a translation.

3.3. Prove that a linear operator on $\mathbb{R}^2$ is a reflection if and only if its eigenvalues are 1 and -1, and the eigenvectors with these eigenvalues are orthogonal.

3.4. Prove that a conjugate of a glide reflection in M is a glide reflection, and that the glide vectors have the same length.

3.5. Write formulas for the isometries (6.3.1) in terms of a complex variable $z = x + iy$.

3.6. **(a)** Let s be the rotation of the plane with angle $\pi/2$ about the point $(1, 1)^t$. Write the formula for s as a product $t_a \rho_\theta$.

(b) Let s denote reflection of the plane about the vertical axis $x = 1$. Find an isometry g such that $grg^{-1} = s$, and write s in the form $t_a \rho_\theta r$.

Section 4 Finite Groups of Orthogonal Operators on the Plane

4.1. Write the product $x^2yx^{-1}y^{-1}x^3y^3$ in the form x^iy^j in the dihedral group D_n.

4.2. **(a)** List all subgroups of the dihedral group D_4, and decide which ones are normal.

(b) List the proper normal subgroups N of the dihedral group D_{15}, and identify the quotient groups D_{15}/N.

(c) List the subgroups of D_6 that do not contain x^3.

4.3. **(a)** Compute the left cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10}.

(b) Prove that H is normal and that D_{10}/H is isomorphic to D_5.

(c) Is D_{10} isomorphic to $D_5 \times H$?

Section 5 Discrete Groups of Isometries

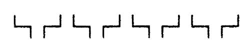
5.1. Let ℓ_1 and ℓ_2 be lines through the origin in $\mathbb{R}^2$ that intersect in an angle π/n, and let r_i be the reflection about ℓ_i. Prove that r_1 and r_2 generate a dihedral group D_n.

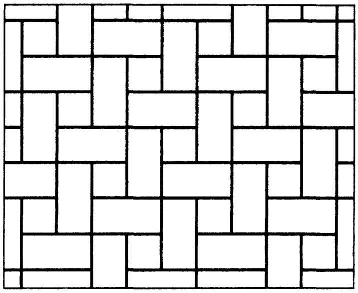
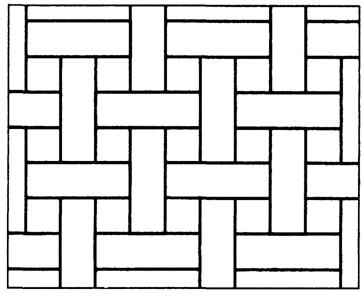
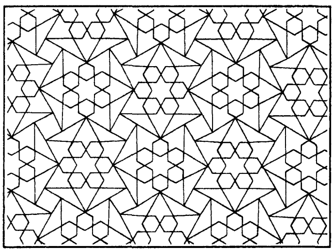
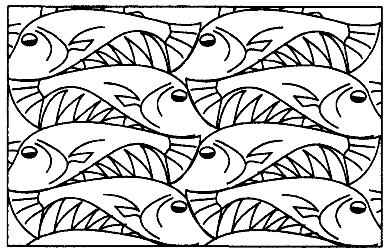
5.2. What is the crystallographic restriction for a discrete group of isometries whose translation group L has the form $\mathbb{Z}a$ with $a \neq 0$?

5.3. How many sublattices of index 3 are contained in a lattice L in $\mathbb{R}^2$?

5.4. Let (a, b) be a lattice basis of a lattice L in $\mathbb{R}^2$. Prove that every other lattice basis has the form $(a', b') = (a, b)P$, where P is a 2×2 integer matrix with determinant ±1.

5.5. Prove that the group of symmetries of the frieze pattern ◁◁◁◁◁◁◁ is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

5.6. Let G be the group of symmetries of the frieze pattern . Determine the point group $\overline{G}$ of G, and the index in G of its subgroup of translations.

*6.4. Classify plane crystallographic groups with point group $D_1 = \{\bar{1}, \bar{r}\}$.

6.5. (a) Prove that if the point group of a two-dimensional crystallographic group G is C_6 or D_6, the translation group L is an equilateral triangular lattice.

(b) Classify those groups.

*6.6. Prove that symmetry groups of the figures in Figure 6.6.2 exhaust the possibilities.

Section 7 Abstract Symmetry: Group Operations

7.1. Let $G = D_4$ be the dihedral group of symmetries of the square.

(a) What is the stabilizer of a vertex? of an edge?

(b) G operates on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

7.2. The group M of isometries of the plane operates on the set of lines in the plane. Determine the stabilizer of a line.

7.3. The symmetric group S_3 operates on two sets U and V of order 3. Decompose the product set $U \times V$ into orbits for the "diagonal action" $g(u, v) = (gu, gv)$, when

(a) the operations on U and V are transitive,

(b) the operation on U is transitive, the orbits for the operation on V are $\{v_1\}$ and $\{v_2, v_3\}$.

7.4. In each of the figures in Exercise 6.3, find the points that have nontrivial stabilizers, and identify the stabilizers.

7.5. Let G be the group of symmetries of a cube, including the orientation-reversing symmetries. Describe the elements of G geometrically.

7.6. Let G be the group of symmetries of an equilateral triangular prism P, including the orientation-reversing symmetries. Determine the stabilizer of one of the rectangular faces of P and the order of the group.

7.7. Let $G = GL_n(\mathbb{R})$ operate on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this operation.

(b) What is the stabilizer of e_1?

7.8. Decompose the set $\mathbb{C}^{2\times 2}$ of 2×2 complex matrices into orbits for the following operations of $GL_2(\mathbb{C})$: **(a)** left multiplication, **(b)** conjugation.

7.9. **(a)** Let S be the set $\mathbb{R}^{m\times n}$ of real $m\times n$ matrices, and let $G = GL_m(\mathbb{R})\times GL_n(\mathbb{R})$. Prove that the rule $(P, Q) * A = PAQ^{-1}$ define an operation of G on S.

(b) Describe the decomposition of S into G-orbits.

(c) Assume that $m \leq n$. What is the stabilizer of the matrix $[I \mid 0]$?

7.10. **(a)** Describe the orbit and the stabilizer of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ under conjugation in the general linear group $GL_n(\mathbb{R})$.

(b) Interpreting the matrix in $GL_2(\mathbb{F}_5)$, find the order of the orbit.

7.11. Prove that the only subgroup of order 12 of the symmetric group S_4 is the alternating group A_4.

Section 8 The Operation on Cosets

8.1. Does the rule $P * A = PAP^t$ define an operation of GL_n on the set of $n\times n$ matrices?

8.2. What is the stabilizer of the coset $[aH]$ for the operation of G on G/H?

8.3. Exhibit the bijective map (6.8.4) explicitly, when G is the dihedral group D_4 and S is the set of vertices of a square.

8.4. Let H be the stabilizer of the index **1** for the operation of the symmetric group $G = S_n$ on the set of indices $\{\mathbf{1}, \ldots, \mathbf{n}\}$. Describe the left cosets of H in G and the map (6.8.4) in this case.

Section 9 The Counting Formula

9.1. Use the counting formula to determine the orders of the groups of rotational symmetries of a cube and of a tetrahedron.

9.2. Let G be the group of rotational symmetries of a cube, let G_v, G_e, G_f be the stabilizers of a vertex v, an edge e, and a face f of the cube, and let V, E, F be the sets of vertices, edges, and faces, respectively. Determine the formulas that represent the decomposition of each of the three sets into orbits for each of the subgroups.

9.3. Determine the order of the group of symmetries of a dodecahedron, when orientation-reversing symmetries such as reflections in planes are allowed.

9.4. Identify the group T' of all symmetries of a regular tetrahedron, including orientation-reversing symmetries.