6.6. Are the matrices $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}$ conjugate elements of the group $GL_2(\mathbb{R})$? Are they conjugate elements of $SL_2(\mathbb{R})$?

6.7. Let H be a subgroup of G, and let g be a fixed element of G. The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, with h in H. Prove that gHg^{-1} is a subgroup of G.

6.8. Prove that the map $A \rightsquigarrow (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.

6.9. Prove that a group G and its opposite group G° (Exercise 2.6) are isomorphic.

6.10. Find all automorphisms of
(a) a cyclic group of order 10, **(b)** the symmetric group S_3.

6.11. Let a be an element of a group G. Prove that if the set $\{1, a\}$ is a normal subgroup of G, then a is in the center of G.

Section 7 Equivalence Relations and Partitions

7.1. Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some g in G is an equivalence relation on G.

7.2. An equivalence relation on S is determined by the subset R of the set $S \times S$ consisting of those pairs (a, b) such that $a \sim b$. Write the axioms for an equivalence relation in terms of the subset R.

7.3. With the notation of Exercise 7.2, is the intersection $R \cap R'$ of two equivalence relations R and R' an equivalence relation? Is the union?

7.4. A relation R on the set of real numbers can be thought of as a subset of the (x, y)-plane. With the notation of Exercise 7.2, explain the geometric meaning of the reflexive and symmetric properties.

7.5. With the notation of Exercise 7.2, each of the following subsets R of the (x, y)-plane defines a relation on the set $\mathbb{R}$ of real numbers. Determine which of the axioms (2.7.3) are satisfied: **(a)** the set $\{(s, s) \mid s \in \mathbb{R}\}$, **(b)** the empty set, **(c)** the locus $\{xy + 1 = 0\}$, **(d)** the locus $\{x^2y - xy^2 - x + y = 0\}$.

7.6. How many different equivalence relations can be defined on a set of five elements?

Section 8 Cosets

8.1. Let H be the cyclic subgroup of the alternating group A_4 generated by the permutation $(1\,2\,3)$. Exhibit the left and the right cosets of H explicitly.

8.2. In the additive group $\mathbb{R}^m$ of vectors, let W be the set of solutions of a system of homogeneous linear equations $AX = 0$. Show that the set of solutions of an inhomogeneous system $AX = B$ is either empty, or else it is an (additive) coset of W.

8.3. Does every group whose order is a power of a prime p contain an element of order p?

8.4. Does a group of order 35 contain an element of order 5? of order 7?

8.5. A finite group contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?

8.6. Let $\varphi: G \to G'$ be a group homomorphism. Suppose that $|G| = 18$, $|G'| = 15$, and that φ is not the trivial homomorphism. What is the order of the kernel?

8.7. A group G of order 22 contains elements x and y, where $x \neq 1$ and y is not a power of x. Prove that the subgroup generated by these elements is the whole group G.

8.8. Let G be a group of order 25. Prove that G has at least one subgroup of order 5, and that if it contains only one subgroup of order 5, then it is a cyclic group.

8.9. Let G be a finite group. Under what circumstances is the map $\varphi: G \to G$ defined by $\varphi(x) = x^2$ an automorphism of G?

8.10. Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

8.11. Let G and H be the following subgroups of $GL_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \right\}, H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

with x and y real and $x > 0$. An element of G can be represented by a point in the right half plane. Make sketches showing the partitions of the half plane into left cosets and into right cosets of H.

8.12. Let S be a subset of a group G that contains the identity element 1, and such that the left cosets aS, with a in G, partition G. Prove that S is a subgroup of G.

8.13. Let S be a set with a law of composition. A partition $\Pi_1 \cup \Pi_2 \cup \cdots$ of S is *compatible* with the law of composition if for all i and j, the product set

$$\Pi_i \Pi_j = \{xy \mid x \in \Pi_i, y \in \Pi_j\}$$

is contained in a single subset Π_k of the partition.

(a) The set $\mathbb{Z}$ of integers can be partitioned into the three sets [Pos], [Neg], [{0}]. Discuss the extent to which the laws of composition $+$ and $\times$ are compatible with this partition.

(b) Describe all partitions of the integers that are compatible with the operation $+$.

Section 9 Modular Arithmetic

9.1. For which integers n does 2 have a multiplicative inverse in $\mathbb{Z}/\mathbb{Z}n$?

9.2. What are the possible values of a^2 modulo 4? modulo 8?

9.3. Prove that every integer a is congruent to the sum of its decimal digits modulo 9.

9.4. Solve the congruence $2x \equiv 5$ modulo 9 and modulo 6.

9.5. Determine the integers n for which the pair of congruences $2x - y \equiv 1$ and $4x + 3y \equiv 2$ modulo n has a solution.

9.6. Prove the *Chinese Remainder Theorem*: Let a, b, u, v be integers, and assume that the greatest common divisor of a and b is 1. Then there is an integer x such that $x \equiv u$ modulo a and $x \equiv v$ modulo b.

Hint: Do the case $u = 0$ and $v = 1$ first.

9.7. Determine the order of each of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ when the matrix entries are interpreted modulo 3.

Section 10 The Correspondence Theorem

10.1. Describe how to tell from the cycle decomposition whether a permutation is odd or even.

10.2. Let H and K be subgroups of a group G.

(a) Prove that the intersection $xH \cap yK$ of two cosets of H and K is either empty or else is a coset of the subgroup $H \cap K$.

(b) Prove that if H and K have finite index in G then $H \cap K$ also has finite index in G.

10.3. Let G and G' be cyclic groups of orders 12 and 6, generated by elements x and y, respectively, and let $\varphi : G \to G'$ be the map defined by $\varphi(x^i) = y^i$. Exhibit the correspondence referred to in the Correspondence Theorem explicitly.

10.4. With the notation of the Correspondence Theorem, let H and H' be corresponding subgroups. Prove that $[G:H] = [G':H']$.

10.5. With reference to the homomorphism $S_4 \to S_3$ described in Example 2.5.13, determine the six subgroups of S_4 that contain K.

Section 11 Product Groups

11.1. Let x be an element of order r of a group G, and let y be an element of G' of order s. What is the order of (x, y) in the product group $G \times G'$?

11.2. What does Proposition 2.11.4 tell us when, with the usual notation for the symmetric group S_3, K and H are the subgroups $\langle y \rangle$ and $\langle x \rangle$?

11.3. Prove that the product of two infinite cyclic groups is not infinite cyclic.

11.4. In each of the following cases, determine whether or not G is isomorphic to the product group $H \times K$.

(a) $G = \mathbb{R}^\times$, $H = \{\pm 1\}$, $K = \{\text{positive real numbers}\}$.

(b) $G = \{\text{invertible upper triangular } 2\times 2 \text{ matrices}\}$, $H = \{\text{invertible diagonal matrices}\}$, $K = \{\text{upper triangular matrices with diagonal entries } 1\}$.

(c) $G = \mathbb{C}^\times$, $H = \{\text{unit circle}\}$, $K = \{\text{positive real numbers}\}$.

11.5. Let G_1 and G_2 be groups, and let Z_i be the center of G_i. Prove that the center of the product group $G_1 \times G_2$ is $Z_1 \times Z_2$.

11.6. Let G be a group that contains normal subgroups of orders 3 and 5, respectively. Prove that G contains an element of order 15.

11.7. Let H be a subgroup of a group G, let $\varphi: G \to H$ be a homomorphism whose restriction to H is the identity map, and let N be its kernel. What can one say about the product map $H \times N \to G$?

11.8. Let G, G', and H be groups. Establish a bijective correspondence between homomorphisms $\Phi : H \to G \times G'$ from H to the product group and pairs (φ, φ') consisting of a homomorphism $\varphi: H \to G$ and a homomorphism $\varphi' : H \to G'$.

11.9. Let H and K be subgroups of a group G. Prove that the product set HK is a subgroup of G if and only if $HK = KH$.

Section 12 Quotient Groups

12.1. Show that if a subgroup H of a group G is not normal, there are left cosets aH and bH whose product is not a coset.

12.2. In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $*$ represents an arbitrary real number. Show that H is a subgroup of GL_3, that K is a normal subgroup of H, and identify the quotient group H/K. Determine the center of H.

12.3. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P that contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.

12.4. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly. Is G/H isomorphic to G?

12.5. Let G be the group of upper triangular real matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, with a and d different from zero. For each of the following subsets, determine whether or not S is a subgroup, and whether or not S is a normal subgroup. If S is a normal subgroup, identify the quotient group G/S.

(i) S is the subset defined by $b = 0$.
(ii) S is the subset defined by $d = 1$.
(iii) S is the subset defined by $a = d$.

Miscellaneous Problems

M.1. Describe the column vectors $(a, c)^t$ that occur as the first column of an integer matrix A whose inverse is also an integer matrix.

M.2. **(a)** Prove that every group of even order contains an element of order 2.
(b) Prove that every group of order 21 contains an element of order 3.

M.3. Classify groups of order 6 by analyzing the following three cases:

(i) G contains an element of order 6.
(ii) G contains an element of order 3 but none of order 6.
(iii) All elements of G have order 1 or 2.

M.4. A *semigroup* S is a set with an associative law of composition and with an identity. Elements are not required to have inverses, and the Cancellation Law need not hold. A semigroup S is said to be generated by an element s if the set $\{1, s, s^2, \dots\}$ of nonnegative powers of s is equal to S. Classify semigroups that are generated by one element.

M.5. Let S be a finite semigroup (see Exercise M.4) in which the Cancellation Law 2.2.3 holds. Prove that S is a group.

***M.6.** Let $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$ be points in k-dimensional space $\mathbb{R}^k$. A *path* from a to b is a continuous function on the unit interval $[0, 1]$ with values in $\mathbb{R}^k$, a function $X: [0, 1] \to \mathbb{R}^k$, sending $t \rightsquigarrow X(t) = (x_1(t), \dots, x_k(t))$, such that $X(0) = a$ and $X(1) = b$. If S is a subset of $\mathbb{R}^k$ and if a and b are in S, define $a \sim b$ if a and b can be joined by a path lying entirely in S.