4.8. **(a)** Prove that the elementary matrices of the first and third types (1.2.4) generate $GL_n(\mathbb{R})$.

(b) Prove that the elementary matrices of the first type generate $SL_n(\mathbb{R})$. Do the 2×2 case first.

4.9. How many elements of order 2 does the symmetric group S_4 contain?

4.10. Show by example that the product of elements of finite order in a group need not have finite order. What if the group is abelian?

4.11. **(a)** Adapt the method of row reduction to prove that the transpositions generate the symmetric group S_n.

(b) Prove that, for $n \geq 3$, the three-cycles generate the alternating group A_n.

Section 5 Homomorphisms

5.1. Let $\varphi: G \to G'$ be a surjective homomorphism. Prove that if G is cyclic, then G' is cyclic, and if G is abelian, then G' is abelian.

5.2. Prove that the intersection $K \cap H$ of subgroups of a group G is a subgroup of H, and that if K is a normal subgroup of G, then $K \cap H$ is a normal subgroup of H.

5.3. Let U denote the group of invertible upper triangular 2×2 matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \rightsquigarrow a^2$. Prove that φ is a homomorphism, and determine its kernel and image.

5.4. Let $f: \mathbb{R}^+ \to \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

5.5. Prove that the $n \times n$ matrices that have the block form $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, with A in $GL_r(\mathbb{R})$ and D in $GL_{n-r}(\mathbb{R})$, form a subgroup H of $GL_n(\mathbb{R})$, and that the map $H \to GL_r(\mathbb{R})$ that sends $M \rightsquigarrow A$ is a homomorphism. What is its kernel?

5.6. Determine the center of $GL_n(\mathbb{R})$.

Hint: You are asked to determine the invertible matrices A that commute with every invertible matrix B. Do not test with a general matrix B. Test with elementary matrices.

Section 6 Isomorphisms

6.1. Let G' be the group of real matrices of the form $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$. Is the map $\mathbb{R}^+ \to G'$ that sends x to this matrix an isomorphism?

6.2. Describe all homomorphisms $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$. Determine which are injective, which are surjective, and which are isomorphisms.

6.3. Show that the functions $f = 1/x$, $g = (x-1)/x$ generate a group of functions, the law of composition being composition of functions, that is isomorphic to the symmetric group S_3.

6.4. Prove that in a group, the products ab and ba are conjugate elements.

6.5. Decide whether or not the two matrices $A = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ are conjugate elements of the general linear group $GL_2(\mathbb{R})$.

6.6. Are the matrices $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}$ conjugate elements of the group $GL_2(\mathbb{R})$? Are they conjugate elements of $SL_2(\mathbb{R})$?

6.7. Let H be a subgroup of G, and let g be a fixed element of G. The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, with h in H. Prove that gHg^{-1} is a subgroup of G.

6.8. Prove that the map $A \rightsquigarrow (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.

6.9. Prove that a group G and its opposite group G° (Exercise 2.6) are isomorphic.

6.10. Find all automorphisms of
(a) a cyclic group of order 10, **(b)** the symmetric group S_3.

6.11. Let a be an element of a group G. Prove that if the set $\{1, a\}$ is a normal subgroup of G, then a is in the center of G.

Section 7 Equivalence Relations and Partitions

7.1. Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some g in G is an equivalence relation on G.

7.2. An equivalence relation on S is determined by the subset R of the set $S \times S$ consisting of those pairs (a, b) such that $a \sim b$. Write the axioms for an equivalence relation in terms of the subset R.

7.3. With the notation of Exercise 7.2, is the intersection $R \cap R'$ of two equivalence relations R and R' an equivalence relation? Is the union?

7.4. A relation R on the set of real numbers can be thought of as a subset of the (x, y)-plane. With the notation of Exercise 7.2, explain the geometric meaning of the reflexive and symmetric properties.

7.5. With the notation of Exercise 7.2, each of the following subsets R of the (x, y)-plane defines a relation on the set $\mathbb{R}$ of real numbers. Determine which of the axioms (2.7.3) are satisfied: **(a)** the set $\{(s, s) \mid s \in \mathbb{R}\}$, **(b)** the empty set, **(c)** the locus $\{xy + 1 = 0\}$, **(d)** the locus $\{x^2y - xy^2 - x + y = 0\}$.

7.6. How many different equivalence relations can be defined on a set of five elements?

Section 8 Cosets

8.1. Let H be the cyclic subgroup of the alternating group A_4 generated by the permutation $(1\,2\,3)$. Exhibit the left and the right cosets of H explicitly.

8.2. In the additive group $\mathbb{R}^m$ of vectors, let W be the set of solutions of a system of homogeneous linear equations $AX = 0$. Show that the set of solutions of an inhomogeneous system $AX = B$ is either empty, or else it is an (additive) coset of W.

8.3. Does every group whose order is a power of a prime p contain an element of order p?

8.4. Does a group of order 35 contain an element of order 5? of order 7?

8.5. A finite group contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?

8.6. Let $\varphi: G \to G'$ be a group homomorphism. Suppose that $|G| = 18$, $|G'| = 15$, and that φ is not the trivial homomorphism. What is the order of the kernel?

8.7. A group G of order 22 contains elements x and y, where $x \neq 1$ and y is not a power of x. Prove that the subgroup generated by these elements is the whole group G.

8.8. Let G be a group of order 25. Prove that G has at least one subgroup of order 5, and that if it contains only one subgroup of order 5, then it is a cyclic group.

8.9. Let G be a finite group. Under what circumstances is the map $\varphi: G \to G$ defined by $\varphi(x) = x^2$ an automorphism of G?

8.10. Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

8.11. Let G and H be the following subgroups of $GL_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \right\}, \quad H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

with x and y real and $x > 0$. An element of G can be represented by a point in the right half plane. Make sketches showing the partitions of the half plane into left cosets and into right cosets of H.

8.12. Let S be a subset of a group G that contains the identity element 1, and such that the left cosets aS, with a in G, partition G. Prove that S is a subgroup of G.

8.13. Let S be a set with a law of composition. A partition $\Pi_1 \cup \Pi_2 \cup \cdots$ of S is *compatible* with the law of composition if for all i and j, the product set

$$\Pi_i \Pi_j = \{xy \mid x \in \Pi_i, y \in \Pi_j\}$$

is contained in a single subset Π_k of the partition.

(a) The set $\mathbb{Z}$ of integers can be partitioned into the three sets [Pos], [Neg], [{0}]. Discuss the extent to which the laws of composition $+$ and $\times$ are compatible with this partition.

(b) Describe all partitions of the integers that are compatible with the operation $+$.

Section 9 Modular Arithmetic

9.1. For which integers n does 2 have a multiplicative inverse in $\mathbb{Z}/\mathbb{Z}n$?

9.2. What are the possible values of a^2 modulo 4? modulo 8?

9.3. Prove that every integer a is congruent to the sum of its decimal digits modulo 9.

9.4. Solve the congruence $2x \equiv 5$ modulo 9 and modulo 6.

9.5. Determine the integers n for which the pair of congruences $2x - y \equiv 1$ and $4x + 3y \equiv 2$ modulo n has a solution.

9.6. Prove the *Chinese Remainder Theorem*: Let a, b, u, v be integers, and assume that the greatest common divisor of a and b is 1. Then there is an integer x such that $x \equiv u$ modulo a and $x \equiv v$ modulo b.

Hint: Do the case $u = 0$ and $v = 1$ first.

9.7. Determine the order of each of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ when the matrix entries are interpreted modulo 3.