OVERTWISTED POSITIVE CONTACT SURGERIES

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ABSTRACT. We study the tightness of positive contact surgery on Legendrian knots in tight contact 3-manifolds. Along with more general results, we give a partial generalisation of a result of Lisca and Stipsicz: if L is a null-homologous Legendrian knot with $tb(L) \leqslant -2$ and $|rot(L)| > 2g(L)-1+tb(L)$, then contact (1)–surgery on L is overtwisted. We also give a condition under which all positive contact surgeries on a Legendrian knot are overtwisted.

1. INTRODUCTION

Given a Legendrian knot L in $(S^3, \xi_{\mathrm{std}})$ with $tb(L) \leqslant -2$, Lisca and Stipsicz proved in [18] that contact (1)–surgery on L has vanishing Heegaard Floer contact invariant. It is natural to ask whether these are indeed overtwisted. We answer this question for a large class of such knots, as well as more generally in an arbitrary contact 3-manifold.

Theorem 1.1. *Let $L \subset (M, \xi)$ be a null-homologous Legendrian knot, where $c_1(\xi)$ is torsion, and $tb(L) \leqslant -2$. Then if*

$$\left|n \cdot rot(L) - (n-1) \cdot tb(L)\right| > n(2g(L)-1) + tb(L)$$

for a positive integer $n < |tb(L)|$, where $g(L)$ is the genus of L, then contact (n)–surgery on L is overtwisted.

In particular, if $tb(L) \leqslant -2$ and $|rot(L)| > 2g - 1 + tb(L)$, then contact (1)–surgery is overtwisted.

Remark 1.2. The necessity of a condition on $rot(L)$ in Theorem 1.1 was shown by Onaran [20]. However, the condition given above might not be the tightest possible condition; even so, the current condition only leaves out a finite set of (tb, rot) pairs for each knot genus g.

Recall that unless $n = 1/k$, there is no unique choice of contact (n)–surgery. However, there is a preferred set of choices (which we call the *natural* surgery), which correspond to inadmissible transverse surgery on the positive transverse push-off of L, see [4]. In this paper, by "contact (n)–surgery", we mean the natural surgery, unless otherwise stated. For a precise definition, see Section 2.

Our second result gives a condition under which *all* positive contact surgeries on L are overtwisted.

Theorem 1.3. *Let $L \subset (M, \xi)$ be a null-homologous Legendrian knot, where $c_1(\xi)$ is torsion, and let $g(L)$ be the genus of L.*

- *If $tb(L) - rot(L) < -2g(L) - 1$, then all natural positive contact surgeries on L are overtwisted.*
- *If, in addition, $tb(L) + rot(L) < -2g(L) - 1$, then all (including non-natural) positive contact surgeries on L are overtwisted.*

The following statement is a natural corollary of Theorem 1.1.

Corollary 1.4. *For every genus g and every positive integer $n \geqslant 2$, there is a positive integer $N_{g,n}$ such that if L is a null-homologous Legendrian knot of genus g and $tb(L) \leqslant -N_{g,n}$, then contact (n)–surgery on L is overtwisted.*

Another result of Lisca and Stipsicz [18] is that contact (1)–surgery on a negative torus knot is overtwisted. Since all negative torus knots satisfy both conditions in Theorem 1.3, we can say the following.

Corollary 1.5. *All (including non-natural) positive contact surgeries on negative torus knots in* $(S^3, \xi_{\mathrm{std}})$ *are overtwisted.*

The figure-eight knot $L \subset (S^3, \xi_{\mathrm{std}})$ with $tb(L) = -3$ and $rot(L) = 0$ is the simplest knot that does not satisfy the conditions of Theorem 1.3. However, the fact that all positive contact surgeries on L are overtwisted can be proved using convex surface theory, see [5].

Looking beyond Lisca and Stipsicz's result, Mark and Tosun [19] have completely characterised when positive contact surgery on $L \subset (S^3, \xi_{\mathrm{std}})$ has non-vanishing Heegaard Floer contact variant. Since this is an algebraic and not contact geometric result, the following natural question is still open.

Question 1.6. *Is the result of positive contact surgery on a Legendrian knot in* S^3 *overtwisted whenever the Heegaard Floer contact class of the surgered manifold vanishes?*

1.1. **Organisation of Paper.** Section 2 gives the necessary background on contact surgery. Section 3 discusses invariants of rationally null-homologous Legendrian knots, which will be needed in the proofs of Theorems 1.1 and 1.3, which can be found in Section 4.

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2. Contact Surgery

We give a description of positive contact surgery, following [15]. We assume basic knowledge of contact structures and Legendrian knots (see [9,10]) as well as some convex surface theory (see [8]).

Given a null-homologous Legendrian knot $L \subset (M, \xi)$, we define contact (r)–surgery on L, for some rational $r > 0$. The surgery coefficient r is understood to be given with respect to the contact framing on L, and so contact (r)–surgery on L results in the manifold given by smooth $(tb(L) + r)$–surgery on K, the underlying smooth knot type of L.

Pick a standard neighbourhood of L that has convex boundary, divided by two dividing curves that give the contact framing of L. Our goal is to remove the interior of the standard neighbourhood, glue in a solid torus to achieve the correct smooth surgery, and then extend the contact structure on the complement of L over the new solid torus such that it restricts to a tight contact structure on the solid torus.

In general, such an extension is not unique. There is a unique tight extension over a solid torus with given convex boundary when the dividing set on the boundary is a pair of curves isotopic to the core of the solid torus, by Kanda [16]. However, there are two minimally twisted tight extensions over $T^2 \times [0, 1]$ with given convex boundary, such that the homology classes of the two pairs of dividing curves on $T^2 \times \{0\}$ and $T^2 \times \{1\}$ form an integral basis for $H_1(T^2)$, by Honda [15] (see there for the precise definition of all terms). Such a contact structure on $T^2 \times I$ is called a *basic slice*. These two extensions can be distinguished by their relative Euler classes, which differ by a sign, and we call them *positive* and *negative*. The terminology can be pinned down by requiring that the result of gluing a negative (resp. positive) basic slice to the complement of a Legendrian knot L is the complement of a negative (resp. positive) stabilisation of L.

Given a rational number $r > 0$, to do contact (r)–surgery, pick a sequence of rational numbers $tb(L) = r_0, \ldots, r_n = tb(L) + r$, where r_{i-1} and r_i are slopes of curves on T^2 that form an integral basis of $H_1(T^2)$ (where p/q corresponds to p meridians and q Seifert longitudes). Let M_0 be the complement of L. For each $i = 1, \ldots, n-1$, construct M_i by gluing a basic slice (either positive or negative) to M_{i-1} along the boundary, such that the new boundary has dividing curves of slope r_i. Finally, construct M_n by gluing a solid torus to M_{n-1} such that the slope $r_n = tb(L) + r$ bounds a disc in the resulting manifold.

Not all such choices give a tight contact structure on the solid torus $M_n \backslash M_0$. However, if we pick the shortest such path such that $r_1 = \infty$ and $r_2 > r_3 > \cdots > r_n$, then this will always result in a tight contact structure on the solid torus. We define the *natural* contact (r)–surgery to be the contact structure created by using the path described above, and only choosing negative basic slices. This is the most well-studied choice, and also is the one that corresponds to inadmissible transverse surgery on a positive transverse push-off of L, see [4].

3. Invariants of Rationally Null-Homologous Legendrian Knots

In this section, we define the rational Thurston–Bennequin $tb_{\mathbb{Q}}(L)$ and the rational rotation number $rot_{\mathbb{Q}}(L)$ of a rationally null-homologous Legendrian knot L, and see how these invariants change under surgery. See [1] for more details and properties.

Given a rationally null-homologous Legendrian knot $L \subset (M, \xi)$, where M is a rational homology sphere, let Σ be a *rational Seifert surface* for L with connected binding. That is, $\partial\Sigma$ is connected and is homologous to $r \cdot [L]$, where r is the smallest positive integer such that $r \cdot [L] = 0 \in H_1(M; \mathbb{Z})$. Given another Legendrian knot L', we define the *rational linking* to be

$$lk_{\mathbb{Q}}(L, L') = \frac{1}{r}\,[\Sigma] \cdot [L'].$$

Consider the framing of the normal bundle of L induced by $\xi|_L$. Let the Legendrian knot L' be a push-off of L in the direction of this framing. Then we define the *rational Thurston–Bennequin* number of L to be

$$tb_{\mathbb{Q}}(L) = lk_{\mathbb{Q}}(L, L').$$

Let $\iota : \Sigma \to M$ be an embedding on the interior of Σ. We choose a trivialisation τ of the pull-back bundle $\iota^*(\xi)$ over Σ. Along $\partial\Sigma$, τ gives an isomorphism of the bundle to $\partial\Sigma \times \mathbb{R}^2$.The tangent vector to $\partial\iota(\Sigma)$ gives a framing of $\xi|_L$, so its pullback v gives a framing of $\iota^*(\xi)$ along $\partial\Sigma$. We define the *rational rotation number* of L to be

$$rot_{\mathbb{Q}}(L) = \frac{1}{r}\,wind_{\tau}(v),$$

where $wind_{\tau}(v)$ measures the winding number of v in $\mathbb{R}^2$ with respect to the trivialisation τ.

A Legendrian knot L in an overtwisted contact manifold is called *loose* if the complement of a standard neighbourhood of L is overtwisted; otherwise, it is called *non-loose*.

Theorem 3.1 (Świątkowski [7], Etnyre [11], Baker–Onaran [2]). *If $L \subset (M, \xi)$ is a rationally null-homologous Legendrian knot such that the complement of a regular neighbourhood of L is tight, then*

$$-|tb_{\mathbb{Q}}(L)| + |rot_{\mathbb{Q}}(L)| \leqslant -\frac{\chi(L)}{r},$$

where r is the order of $[L]$ in $H_1(M; \mathbb{Z})$, and $\chi(L)$ is the Euler characteristic of a rational Seifert surface for L.

Remark 3.2. In fact, Baker and Etnyre showed [1] that if the contact structure is tight, the rational homotopy invariants of a rationally null-homologous Legendrian knot satisfy $tb_{\mathbb{Q}}(L) + |rot_{\mathbb{Q}}(L)| \leqslant -\chi(L)/r$. This gives in some cases a better inequality than that from Theorem 3.1, but does not improve the results of this paper.

The following lemma has been proved by Lisca, Ozsváth, Stipsicz, and Szabó [17] and Geiges and Onaran [13] for surgeries in $(S^3, \xi_{\rm std})$. We extend it to surgeries in a more general contact manifold.

Proposition 3.3. *Let $L_0 \cup \cdots \cup L_n \subset (M, \xi)$ be a collection of null-homologous Legendrian knots, where $c_1(\xi)$ is torsion. Perform contact (± 1)–surgery on L_i, for $i = 1, \ldots, n$ (the sign need not be the same for each i), and let a_i be the smooth surgery coefficient. Assume that the resulting manifold (M', ξ') has the same rational homology as M. Let $N = (N_{ij})$ for $1 \leqslant i, j \leqslant n$ be the matrix given by*

$$N_{ij} = \begin{cases} a_i & i = j, \\ lk(L_i, L_j) & i \neq j, \end{cases}$$

and let $N_0 = ((N_0)_{ij})$ *for* $0 \leqslant i, j \leqslant n$ *be the matrix given by*

$$(N_0)_{ij} = \begin{cases} 0 & i = j = 0, \\ a_i & i = j \geqslant 1, \\ lk(L_i, L_j) & i \neq j. \end{cases}$$

Then the rational classical invariants for L*, the image of* L_0 *in* (M', ξ')*, are*

$$tb_{\mathbb{Q}}(L) = tb(L_0) + \frac{\det N_0}{\det N},$$

and

$$rot_{\mathbb{Q}}(L) = rot(L_0) - \left\langle \begin{pmatrix} rot(L_1) \\ \vdots \\ rot(L_n) \end{pmatrix}, N^{-1} \begin{pmatrix} lk(L_0, L_1) \\ \vdots \\ lk(L_0, L_n) \end{pmatrix} \right\rangle.$$

Proof. We refer the reader to [13, Lemma 2] for the proof of the formula for $tb_{\mathbb{Q}}(L)$, which goes through essentially unchanged; we give here the proof of the formula for $rot_{\mathbb{Q}}(L)$, explained in full to allow for the differences to be comprehensible. For each $i = 0, \ldots, n$, let λ_i and μ_i be the Seifert framing and meridian respectively for L_i in M. Because each L_i is null-homologous, we can conclude that

$$H_1(M' \backslash L) \cong H_1(M) \oplus \Big(\mathbb{Z}\langle \mu_0 \rangle \oplus \cdots \oplus \mathbb{Z}\langle \mu_n \rangle\Big) / \langle a_i \mu_i + \sum_{\substack{j=0 \\ j \neq i}}^{n} lk(L_i, L_j)\mu_j = 0, i = 1, \ldots, n \rangle.$$

From the Mayer-Vietoris sequence of $M' = M' \backslash L \cup L$, we get the short exact sequence

$$0 \to \mathbb{Z}\langle \mu_0 \rangle \oplus \mathbb{Z}\langle \lambda_0 \rangle \to H_1(M' \backslash L) \oplus H_1(L) \to H_1(M') \to 0.$$

Note that $\mathbb{Z}\langle \lambda_0 \rangle \to H_1(L)$ is an isomorphism, and μ_0 maps to 0 in $H_1(L)$. Note also that the $H_1(M)$ summand in $H_1(M' \backslash L)$ maps isomorphically onto the $H_1(M)$ summand in $H_1(M')$, and the other summands of $H_1(M' \backslash L)$ map into the other summands of $H_1(M')$. Thus we can get the short exact sequence

$$0 \to \mathbb{Z}\langle \mu_0 \rangle \to H_1(M' \backslash L)/H_1(M) \to H_1(M')/H_1(M) \to 0.$$

Since $H_1(M'; \mathbb{Q}) = H_1(M; \mathbb{Q})$, the preceding exact sequence considered with rational coefficients implies that the residue of $\mathrm{PD}\, c_1(\xi', L)$ in $H_1(M' \backslash L; \mathbb{Q})/H_1(M; \mathbb{Q})$ is some rational multiple of μ_0.

To get a formula for $\mathrm{PD}\, c_1(\xi, \bigcup_{i=0}^{n} L_i)$, we start with a non-zero vector field v over L_i. Given Seifert surfaces $\Sigma_0, \ldots, \Sigma_n$ for $L_0, \ldots, L_n$ in M, we extend v over Σ_i such that there are $rot(L_i)$ zeroes over Σ_i. Finally, we extend over the rest of M. The zero set of v tells us that

$$\mathrm{PD}\, c_1(\xi, \bigcup_{i=0}^{n} L_i) = \sum_{i=0}^{n} rot(L_i)\mu_i + x,$$

where x is the push-forward of some class in $H_1(M)$ that by construction does not intersect Σ_i.

We claim that we can construct a rational Seifert surface Σ for L in M' such that $x \cdot [\Sigma] = 0$. We then calculate that in (M', ξ'), if L is order r in $H_1(M')$, then

$$r \cdot rot_{\mathbb{Q}}(L) = \mathrm{PD}\, c_1(\xi', L) \cdot [\Sigma].$$

Notice that $\mathrm{PD}\, c_1(\xi', L)$ is the push-forward of $\mathrm{PD}\, c_1(\xi, \bigcup_{i=0}^{n} L_i)$, and $x \cdot [\Sigma] = 0$. Thus with rational coefficients, the only free part left that could act non-trivially on Σ is generated by μ_0, and since $\mu_0 \cdot [\Sigma] = r$, it must be $rot_{\mathbb{Q}}(L)\mu_0$.

With rational coefficients, the summand of PD $c_1(\xi, \bigcup_{i=0}^{n} L_i)$ corresponding to the μ_i can be written as an element of the $\mathbb{Q}$ summand of $H_1(M';\mathbb{Q})$ generated by μ_0. Thus we have the equation

$$\sum_{i=0}^{n} rot(L_i)\mu_i = rot_{\mathbb{Q}}(L)\mu_0$$

in $H_1(M\backslash L_0;\mathbb{Q})$. Note that the surgery gives a cobordism $X : M \to M'$, where $H_2(X) = H_2(M) \oplus \mathbb{Z}^n$ and $H_2(X, M) = \mathbb{Z}^n$. Thus the long-exact sequence of the pair (X, M) gives

$$H_2(M) \to H_2(X) \to H_2(X, M),$$

where the first map is an isomorphism into the $H_2(M)$ summand of $H_2(X)$, and the second map is 0 on the $H_2(M)$ summand, and acts as the matrix N on the $\mathbb{Z}^n$ summand. Thus we see that N is an injective map, and thus with rational coefficients, we can invert it. The formula for $rot_{\mathbb{Q}}(L)$ then follows.

Let $\widetilde{\Sigma}_i$ be Σ_i minus the interior of $N(L_0)\cup\cdots N(L_n)$. To prove the claim, we construct a surface $\widetilde{\Sigma}$ in M in a neighbourhood of $\widetilde{\Sigma}_0 \cup\cdots\cup \widetilde{\Sigma}_n$ such that its image in M' can be capped off (by meridians of the surgery duals to $L_1,\ldots,L_n$) to a rational Seifert surface Σ for L. Since $x\cdot[\Sigma_i]$, we have that $x\cdot[\widetilde{\Sigma}] = 0$ as well.

First note that for every positive intersection of L_i and Σ_j, the intersection of $\partial N(L_i)$ with $\widetilde{\Sigma}_j$ is $-\mu_i$, a meridian that in M links L_i once negatively, see FIgure 1. We start with r copies of $\widetilde{\Sigma}_0$, and we would like to pick $|k_i|$ copies of $\widetilde{\Sigma}_i$, $i = 1,\ldots,n$, such that

$$k_i\lambda_i - \left(r\cdot lk(L_0, L_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \cdot lk(L_i, L_j) \right)\mu_i = k_i\left(\lambda_i + a_i\mu_i\right).$$

If $k_i < 0$, we reverse the orientation of $\widetilde{\Sigma}_i$. This system of equations corresponds to the intersection of the collection of surfaces with $\partial N(L_i)$, for each i. Comparing the coefficients of μ_i in each equation, we see that

$$\sum_{i=1}^{n} k_i \cdot lk(L_i, L_j) = -r\cdot lk(L_0, L_i),$$

where we define $lk(L_i, L_i)$ to be a_i. Notice that this is the same as the equation

$$N\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} -r\cdot lk(L_0, L_1) \\ \vdots \\ -r\cdot lk(L_0, L_n) \end{pmatrix}.$$

Since N is invertible, we can solve for k_i. We claim that each k_i is an integer. To see this, first note that the order r of L is the order of $[L] = \sum_{i=1}^{n} lk(L_0, L_i)\mu_i$ in $H_1(M';\mathbb{Z})$. Since

$$-r\cdot\sum_{i=1}^{n} lk(L_0, L_i)\mu_i = 0$$

in $H_1(M';\mathbb{Z})$, we know that it is a sum of the relations

$$a_i\mu_i + \sum_{\substack{j=0 \\ j\neq i}}^{n} lk(L_i, L_j)\mu_j.$$

Putting these quantities in vector form, this is equivalent to

$$\begin{pmatrix} -r \cdot lk(L_0, L_1) \\ \vdots \\ -r \cdot lk(L_0, L_n) \end{pmatrix}$$

being an integer linear combination of the columns of N, where the coefficients of the linear combination are exactly k_i.

The boundary of the 2-complex given by the union of r copies of $\widetilde{\Sigma}_0$ and k_i copies of $\widetilde{\Sigma}_i$, $i = 1, \dots, n$ is homologous to an (r, s) curve on $\partial N(L)$, for $s = \sum_{i=0}^{n} k_i \cdot lk(L_0, L_i)$, and k_i copies of a $(1, a_i)$ curve on $\partial N(L_i)$. Thus we can find some smooth embedded surface $\widetilde{\Sigma}$ in a neighbourhood of $\widetilde{\Sigma}_0 \cup \cdots \cup \widetilde{\Sigma}_n$ with boundary given by the oriented resolution of the boundary of the 2-complex. The boundary components of $\widetilde{\Sigma}$ on $\partial N(L_i)$, $i = 1, \dots, n$, bound discs in M', and capping off these components gives a rational Seifert surface Σ for L in M'. □

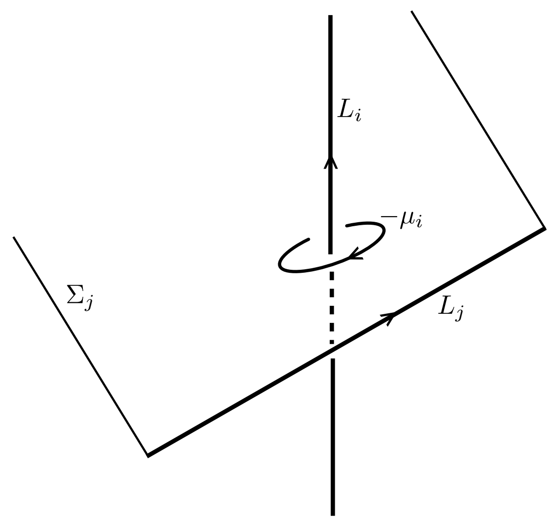

FIGURE 1. When L_i and L_j link positively, the intersection of Σ_j with the boundary of a neighbourhood of L_i is $-\mu_i$.

Remark 3.4. If $c_1(\xi)$ is non-torsion, then rotation numbers may depend on the relative homology class of the Seifert surfaces that are chosen. Thus, given Seifert surfaces $\Sigma_0, \dots, \Sigma_n$, and using $rot(L_i, \Sigma_i)$ in the formulae, the same proof will calculate $rot_{\mathbb{Q}}(L, \Sigma)$, where Σ is constructed from $\Sigma_0, \dots, \Sigma_n$ as in the proof. However, for clarity, we will state all our results in the context of $c_1(\xi)$ torsion.

Remark 3.5. The proof of the formula for $tb_{\mathbb{Q}}(L)$ is entirely topological. Thus, if we consider the contact surgery diagram as a smooth surgery diagram, and perform Kirby calculus moves on the diagram, then using the M and M_0 from the new diagram will still give the correct value for $tb_{\mathbb{Q}}(L)$. The calculations for $rot_{\mathbb{Q}}(L)$, however, are contact geometric in nature, and so must respect the contact surgery diagram chosen.

4. OVERTWISTED SURGERIES

In this section, we prove Theorems 1.1 and 1.3. We start by describing the approach to both proofs.

Given a Legendrian $L \subset (M,\xi)$, consider the core of the surgery torus L^* in $(M_{tb(L)+1}(L), \xi_{(1)}(L))$, the result of contact (1)–surgery on L. This is naturally a Legendrian knot, and the complement of a standard neighbourhood of L^* is naturally identified with the complement of a standard neighbourhood of L in (M,ξ). We denote the result of a single negative (resp. positive) stabilisation of L^* by L^*_- (resp. L^*_+). It is not hard to see that a curve giving the contact framing of $L^*_\pm$ is isotopic in the complement of L to a meridian of L.

Thus, the complement of a standard neighbourhood of L^*_- (resp. L^*_+) with convex boundary is the manifold obtained from the complement of L by attaching a negative (resp. positive) bypass to get meridional dividing curves. We claim that if these contact manifolds with convex boundary are overtwisted, then all positive contact surgeries on L are overtwisted. This follows from the description of positive contact surgery in Section 2: the first step is to add a bypass layer to get to the meridional slope, and then to glue on further basic slices and a solid torus. Thus, the contact manifold with meridional dividing curves on the boundary embeds into the final result of contact surgery, and so if it is overtwisted, it follows that the surgered manifold is also overtwisted. If we can show that the complement of a standard neighbourhood of L^*_- is overtwisted, then we can show that all natural contact surgeries on L are overtwisted (and in fact any surgery that begins with a negative basic slice). If, in addition, the complement of a standard neighbourhood of L^*_+ is overtwisted, then it follows that all positive contact surgeries on L (regardless of choices) are overtwisted.

Our tool to show that the complement of a standard neighbourhood of a Legendrian knot is overtwisted is Theorem 3.1, and we use Proposition 3.3 to calculate the relevant invariants.

We first prove Theorem 1.3. Our proof was inspired by (and runs similarly to) [2, Theorem 4.1.8].

Proof of Theorem 1.3. Let K be the smooth knot type of L. Consider contact (1)–surgery on L, and let L^* be the surgery dual. This surgery is smoothly equivalent to smooth $(tb(L)+1)$–surgery on K. Consider a Legendrian push-off L_0 of L. This is smoothly a $(1, tb(L))$ curve on the boundary of a neighbourhood of L, where the longitude is given by the Seifert framing. In particular, L_0 is parallel to the dividing curves on the convex boundary of a standard neighbourhood of L. Thus L_0^*, the image of L_0 after surgery on L, is still parallel to the dividing curves on the boundary of a standard neighbourhood of L^*, and thus is Legendrian isotopic to L^*. Hence we conclude that

$$\chi(L_0^*) = \chi(L^*) = \chi(L) = 1 - 2g(L).$$

We use Proposition 3.3 to work out the rational Thurston–Bennequin and rotation numbers of L_0^*. We have

$$N = (tb(L)+1)$$

and

$$N_0 = \begin{pmatrix} 0 & tb(L) \\ tb(L) & tb(L)+1 \end{pmatrix}.$$

Thus since $tb(L_0) = tb(L)$, $rot(L_0) = rot(L)$, and $lk(L_0, L) = tb(L)$, we calculate

$$\begin{aligned} tb_{\mathbb{Q}}(L_0^*) &= tb(L_0) + \frac{\det N_0}{\det N} \\ &= tb(L) - \frac{tb(L)^2}{tb(L)+1} \\ &= \frac{tb(L)}{tb(L)+1}, \\ rot_{\mathbb{Q}}(L_0^*) &= rot(L_0) - rot(L) \cdot N^{-1} lk(L_0, L) \\ &= rot(L) - rot(L) \cdot \left(\frac{1}{tb(L)+1}\right)(tb(L)) \\ &= \frac{rot(L)}{tb(L)+1}. \end{aligned}$$

Consider L_+^* and L_-^*, the positive and negative stabilisations of L_0^*. As in the discussion preceding the theorem, the complement of $L_\pm^*$ is exactly the complement of L in (M,ξ) with a positive or negative basic slice added to the boundary to take the dividing curves of the boundary torus to meridional curves. Thus, to show that the latter contact manifolds are overtwisted, we show that $L_\pm^*$ are loose under the hypotheses of the theorem, and then we are done, by the discussion preceding the theorem.

Assume that $tb(L) - rot(L) < -2g(L) - 1$. We will use Theorem 3.1 to prove that L_-^* is loose. We see that

$$tb_{\mathbb{Q}}(L_-^*) = \frac{tb(L)}{tb(L)+1} - 1 = \frac{-1}{tb(L)+1},$$
$$rot_{\mathbb{Q}}(L_-^*) = \frac{rot(L)}{tb(L)+1} - 1 = \frac{rot(L) - tb(L) - 1}{tb(L)+1}.$$

Plugging these into Theorem 3.1, our assumption gives us that

$$-|tb_{\mathbb{Q}}(L_-^*)| + |rot_{\mathbb{Q}}(L_-^*)| = \frac{|tb(L) - rot(L) + 1| - 1}{|tb(L)+1|} > \frac{2g(L)-1}{|tb(L)+1|} = -\frac{\chi(L_-^*)}{|tb(L)+1|},$$

and so L_-^* is loose, by Theorem 3.1.

If we assume now that we also have $tb(L) + rot(L) < -2g(L) - 1$, similar analysis to the above will show that L_+^* is also loose. Then all positive contact surgeries on L (regardless of choices) are overtwisted, as explained in the discussion preceding the theorem. □

Now we turn to the proof of Theorem 1.1, where we look at (natural) contact (n)-surger,y for positive integers n. Note, though, that there is only one other way to do contact (n)–surgery, and that choice is equivalent to the natural contact (n)–surgery on $-L$, which is L with its orientation reversed.

Proof of Theorem 1.1. Let L be a Legendrian knot with $tb(L) = t$ and $rot(L) = r$. Let L_1 and L_2' be push-offs of L. Stabilise L_2' once negatively to get L_2, and let $L_3, \ldots, L_n$ be push-offs of L_2. By [6], contact (n)–surgery on L is equivalent to contact (1)–surgery on L_1 and contact (-1)–surgery on $L_2, \ldots, L_n$. Let L^* be the image of L after the surgeries on $L_1, \ldots, L_n$. According to Remark 3.5, any smooth surgery diagram smoothly equivalent to the original one can be used to calculate $tb_{\mathbb{Q}}(L^*)$. Thus, we can assume we are doing smooth $(t+n)$–surgery on a single knot K that has linking t with L. Thus we can set

$$N = (t+n) \text{ and } N_0 = \begin{pmatrix} 0 & t \\ t & t+n \end{pmatrix}.$$

So we calculate that

$$tb_{\mathbb{Q}}(L^*) = t + \frac{\det N_0}{\det N} = t - \frac{t^2}{t+n} = \frac{tn}{t+n}$$

In order to calculate $rot_{\mathbb{Q}}(L^*)$, however, we must use the contact surgery diagram that we started with. Thus we have the $n \times n$ matrix given by

$$N = \begin{pmatrix} t+1 & t & t & \cdots & t & t \\ t & t-2 & t-1 & \cdots & t-1 & t-1 \\ t & t-1 & t-2 & \cdots & t-1 & t-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t & t-1 & t-1 & \cdots & t-2 & t-1 \\ t & t-1 & t-1 & \cdots & t-1 & t-2 \end{pmatrix}.$$

It can be verified that its inverse is given by

$$N^{-1} = \frac{1}{t+n}\begin{pmatrix} n-(n-1)t & t & t & \cdots & t & t \\ t & 1-n-t & 1 & \cdots & 1 & 1 \\ t & 1 & 1-n-t & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t & 1 & 1 & \cdots & 1-n-t & 1 \\ t & 1 & 1 & \cdots & 1 & 1-n-t \end{pmatrix}.$$

We can then calculate that

$$\begin{pmatrix} r \\ r-1 \\ r-1 \\ \vdots \\ r-1 \end{pmatrix} \cdot N^{-1} \begin{pmatrix} t \\ t \\ t \\ \vdots \\ t \end{pmatrix} = \frac{1}{t+n} \begin{pmatrix} r \\ r-1 \\ r-1 \\ \vdots \\ r-1 \end{pmatrix} \cdot \begin{pmatrix} tn \\ -t \\ -t \\ \vdots \\ -t \end{pmatrix} = \frac{(r+n-1)\cdot t}{t+n}.$$

Finally, we can conclude that

$$rot_{\mathbb{Q}}(L^*) = r - \frac{(n+r-1)\cdot t}{t+n} = \frac{rn - tn + t}{t+n}.$$

We now consider k positive or negative stabilisations $L^*_{\pm k}$ of L^*, and plug the Thurston–Bennequin number and rotation number of $L^*_{\pm k}$ into Theorem 3.1 to show that that the complement of $L^*_{\pm k}$ is overtwisted, and hence that the result of surgery on L is overtwisted.. Let L^*_k be the k-fold stabilisation of L^* with stabilisation sign equal to the sign of $rot_{\mathbb{Q}}(L^*)$ (with any choice if the rotation vanishes). Then for large k,

$$\begin{aligned} -|tb_{\mathbb{Q}}(L^*_k)| + |rot_{\mathbb{Q}}(L^*_k)| &= |rot_{\mathbb{Q}}(L^*)| + k - |tb_{\mathbb{Q}}(L^*) - k| \\ &= \frac{|rn - tn + t|}{|t+n|} + k - \left(k - \frac{|tn|}{|t+n|}\right) \\ &= \frac{|rn - tn + t| + |tn|}{|t+n|}. \end{aligned}$$

The first equality is true because the sign of the stabilisation is chosen to agree with the sign of $rot_{\mathbb{Q}}(L^*)$, and the second equality is true because k is large and $tb_{\mathbb{Q}}(L^*)$ is positive.

We now need to work out the genus of L^*_k, *ie.* the genus of L^*, in order to calculate $\chi(L^*)$. We see that in M, L is a $(1,t)$ cable of K, the single knot on which we performed smooth $(t+n)$–surgery above to get the same manifold as contact surgery on $L_1, \ldots, L_n$ (an (r,s)-cable is a cable composed of r longitudes and s meridians). Although L is smoothly isotopic to K, in general, the image L^* of L in $M_{t+n}(K)$ is not isotopic to K^*, the surgery dual knot to K; if $n = 1$, then it is true that a push-off of a Legendrian knot gives a framing to the surgery dual to contact (1)–surgery on the original Legendrian (and this is also true for contact (-1)–surgery), but this is false for general n. We claim that L^* is in fact a $(-n,1)$-cable of K^*. This can be seen by calculating the image of the cable under the map gluing the surgery torus into $M \backslash N(K)$, in the coordinate system where the longitude of K^* is isotopic to a meridian of K.

$$\begin{pmatrix} 0 & 1 \\ 1 & t+n \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} -t-n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} -n \\ 1 \end{pmatrix}.$$

Letting $m = \gcd(n, |t+n|)$, we see that this knot has order $|t+n|/m$ in $H_1(M_{t+n}(K);\mathbb{Z})$. Note also, that the boundary of the Seifert surface traces a $\binom{-t-n}{1} = \binom{|t+n|}{1}$ curve on the boundary of a neighbourhood of K^*. So we write

$$\frac{|t+n|}{m} \cdot \begin{pmatrix} -n \\ 1 \end{pmatrix} = \frac{-n}{m} \cdot \begin{pmatrix} |t+n| \\ 1 \end{pmatrix} - \frac{t}{m} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where on the right, the first summand is copies of the Seifert surface, and the second summand is copies of the meridian of K^*. Thus, a rational Seifert surface for L^* is composed of n/m copies of the rational Siefert surface Σ for K^* and $|t|/m$ copies of a meridional compressing disc for K^*, with bands corresponding to the intersections of $\left(\begin{smallmatrix}-n\\1\end{smallmatrix}\right)$ with the $|t|/m$ meridians. So

$$\chi(L^*) = \frac{n}{m}\chi(\Sigma) + \frac{|t|}{m} - \frac{|nt|}{m} = \frac{n(1-2g)+nt-t}{m},$$

since $t < 0$ and $n > 0$. Thus if

$$\frac{|rn - tn + t| + |tn|}{|t+n|} > -\frac{\chi(L_k^*)}{|t+n|/m} = \frac{n(2g-1)+t-nt}{|t+n|},$$

then L_k^* is loose and the contact structure on $M_{t+n}(K)$ is overtwisted. Then our inequality is equivalent to requiring

$$|rn - tn + t| > n(2g-1) + t.$$

Hence under our hypotheses, L_k^* is loose as required. □

In contrast to the above results, we present an infinite family of Legendrian knots with arbitrarily low maximum Thurston–Bennequin number that admit tight positive contact surgeries.

Proposition 4.1. *For every positive integer n, there is an infinite family of null-homologous Legendrian knots in $(S^3, \xi_{\mathrm{std}})$ with $tb = -n$, such that contact (n)–surgery on each knot is tight.*

Proof. We look for Legendrian knots L that are smoothly slice (*ie.* $g_4 = 0$), as this implies that $\tau = \epsilon = 0$, by [14], and such that $tb(L) - rot(L) = 2\tau - 1 = -1$. If $tb(L) = -n$, then [19, Theorem 1.2] says that contact (n)–surgery is tight (and the resulting contact manifold has non-vanishing Heegaard Floer contact invariant).

Consider the slice knot 9_{46}, which has a Legendrian representative L_1 with $tb = -1$ and $rot = 0$, and the slice knot 8_{20}, which has a Legendrian representative L_2 with $tb = -2$ and $rot = -1$ (information on both knots is from [3]). Now consider the Legendrian knots

$$L_{m,n} = \left(\overset{m}{\#} L_1\right) \# \left(\overset{n}{\#} L_2\right)$$

for non-negative integers m and n, where the connected sum of 0 objects is taken to be the standard Legendrian unknot. Under connected sum, the slice genus is additive, so $K_{m,n}$ is slice for all m, n. According to [12], the rotation number of Legendrian knots under connected sum is additive, and the Thurston–Bennequin number adds like

$$\overline{tb}(L \# L') = \overline{tb}(L) + \overline{tb}(L') + 1.$$

Thus

$$tb(L_{m,n}) = -n - 1 \text{ and } rot(L) = -n.$$

Then, the infinite family $\{L_{m,n-1}\}_m$ satisfies the desired requirements. □

An earlier version of this paper asked the following question: if L is a null-homologous Legendrian knot with $tb(L) \leqslant -2$, then is contact (n)–surgery on L overtwisted for any positive integer $n < |tb(L)|$? Since then, Onaran [20] has given infinitely many examples giving a negative answer to the original question.

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